LEAST STRATIFICATIONS AND CELL-STRUCTURED OBJECTS IN GEOMETRIC MODELLING

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Results from the study of O-minimal structures are used to formalise fundamental objects and operations for geometric modelling kernels. O-minimal structures provide a more general setting than the traditional semialgebraic sets; they allow semianalytic sets such as screw-threads to be modelled accurately. O-minimality constrains the class of sets to ensure the retention of the key finitary properties necessary for many geometric operations. The formalism is independent of *internal* representations and implementations, in the spirit of the mixed-dimension cellular objects of the Djinn API (Application Programming Interface) [2].

Topological stratification is used to provide an object with a cellular structure consisting of a finite set of piecewise-smooth manifold cells. The main contribution of the paper is an existence proof for least stratifications (of sets in O-minimal structures). Set-like combination of two cell-structured objects requires a common partition of space from which a cell-structured result can be obtained. The existence of least stratifications facilitates the definition of combination operators that provide unique results.

Keywords: Geometric modelling, geometric kernel, semianalytic set, subanalytic set, stratification, cellular structure, O-minimal structure, topology.

1 Introduction

Djinn [2] is a representation-independent application programming interface (API) to “open kernel” geometric modelling systems that underpin CAD and other geometry based applications. In contrast to current CAD systems, the Djinn interface eliminates application dependence on the implementation's representation. Djinn objects have a structure of disjoint cells of various dimensions and Djinn operations include “set-like” combinations of objects that deal with cellular structure, heterogeneous dimensionality, and orientation. This object structure admits, for example, laminated objects and objects with cracks.

Mathematical foundations for Djinn like systems are found in [11] whilst the cell-structured objects of Djinn and “set-like”operations on them are formalised in [9] and [10] respectively. These papers resolve some ambiguities and imprecision in the definitions of [2], and show that significant complexity arises from decisions not essential for geometric modelling purposes. In particular, the use of semianalytic sets implies the need for cells with infinitely many connected components. This paper establishes a key result to justify the adoption of an O-minimal structure such as the finitely subanalytic sets [8]. It is used without proof in [8] to define the Djinn* modifications to the Djinn interface which achieve closure under its objects and operators without sacrificing essential geometric modelling requirements. Djinn* object definitions are significantly different from those of Djinn from a mathematical viewpoint but not from a user viewpoint.

By generalising the semialgebraic sets (the basis of most current geometric modelling systems) to the semianalytic sets, Djinn extends the variety of shapes that can be modelled accurately (e.g. screwthreads). Unfortunately, semianalytic sets lack important finitary properties and thus computationally important properties. Studies of O-minimal structures (see for example [20]) provide new generalisations of the semialgebraic sets. For example, the finitely subanalytic sets [19] form an O-minimal structure that includes the semialgebraic sets and sufficient semianalytic sets for geometric modelling (e.g. they include bounded regions of the sine curve but not the complete unbounded sine curve).

O-minimal structures provide a key concept for geometric modelling because such structures are *collections of sets generated by closure under essential operations which remain within the realms of analytic geometry*. Such collections do not contain sets of points beyond geometric interpretation, e.g. as space filling curves and other sets which are not topologically *tame*. Since it is known that inclusion of a set with infinitely many components allows such *untame* sets to be generated, O-minimal structures only contain sets with finite numbers of connected components.

A major advantage of O-minimal structures is that important geometric results can be established which cover a wide variety of different classes of set. This paper deals with the properties of general O-minimal structures that are relevant to geometric modelling rather than those of specific classes of set. Despite the choice of finitely subanalytic sets by [8], the best choice of O-minimal structure for geometric modelling is unclear, particularly as new structures are still being discovered and explored.

Any set definable in an O-minimal structure admits a decomposition into finitely many cells. In this paper a distinction is made between inductively defined topological **Icells** (the cells of [20] used to prove such decomposition results) and the more general **Dcells** introduced in this paper to provide structure in the objects of geometric modelling systems. An important contribution of this paper is an existence proof for a least stratification of any definable closed set into a finite set of Dcells that are compatible with a given finite collection of definable subsets. This proof ensures that a (necessarily unique) least stratification relative to a finite collection of space-partitions (such as a collection of Djinn* objects) can be constructed. This is a prerequisite for the definition of operations such as "union" and "intersection" on Djinn* objects that give unique results. The proof depends on cellular decomposition results for O-minimal structures.

The next section formally defines O-minimal structures, Icells and the decomposition of definable sets. Section 3 discusses stratifications and Dcells and section 4 proves the key result about least stratifications. Djinn* objects and operations are described in section 5 and the final section discusses conclusions and related work.

2 O-minimal Structures

O-minimal structures have been studied in model theory as a natural (logical) generalisation of semialgebraic sets. This section provides some basic definitions and properties subsequently used to derive results that are useful in geometric modelling.

Semialgebraic Sets: A set in $\mathbb{R}^n$ $(n \geq 0)$ is *semialgebraic* if it is a finite union of sets of the form $\{x \mid f_1(x) = f_2(x) = \ldots = f_k(x) = 0,\ g_1(x) > 0, \ldots, g_l(x) > 0\}$, where the f_i and g_j are polynomial functions $\mathbb{R}^n \to \mathbb{R}$.

Semialgebraic sets are used as a basis for geometric modelling systems because they have many useful properties. For example,

- every semialgebraic set has a finite number of connected components, each of which is also semialgebraic,
- the closure, interior, boundary and frontier of a semialgebraic set is also semialgebraic,
- every semialgebraic set can be triangulated and stratified into real analytic manifolds, and
- semialgebraic sets are closed under projection (Tarski-Seidenberg theorem).

The last property is important because it ensures that any set constructed from semialgebraic sets using first-order logic is also semialgebraic. For example, if each point $x = \langle x_1, \ldots, x_{n+1} \rangle$ of a semialgebraic set A satisfies the predicate $\Phi(x)$, the predicate $\exists x_{n+1} \Phi(x)$ defines $\pi(A)$ where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a projection. Since the complement of a semialgebraic set is also semialgebraic and is defined by the negation of its predicate, we also have closure under universal quantification because $\forall x \Psi \equiv \neg \exists x (\neg \Psi)$. Most of the important properties of semialgebraic sets can be established from these elementary logical properties.

The semialgebraic sets generalise to the semianalytic and subanalytic sets:

Semianalytic Sets: A set A in $\mathbb{R}^n$ $(n \geq 0)$ is *semianalytic at a point* $x \in \mathbb{R}^n$ if the intersection of A with some neighbourhood U of x is a finite union of sets of the form:
$\{x \mid f_1(x) = f_2(x) = \ldots = f_k(x) = 0,\ g_1(x) > 0, \ldots, g_l(x) > 0\}$
where the f_i and g_j are analytic functions $U \to \mathbb{R}$.

A set A is said to be *semianalytic in* $\mathbb{R}^n$ (or sometimes just *semianalytic*) if it is semianalytic at every point* in $\mathbb{R}^n$. A useful example of a non-semianalytic set is the *topologist's curve* $\{\langle x, \sin \frac{\pi}{x} \rangle \mid x > 0\} \subseteq \mathbb{R}^2$ discussed in [8].

* Only points in the closure of A need be considered, but it is insufficient to consider points in A alone.

Subanalytic Sets: The *subanalytic sets* are the smallest class that contains all the semianalytic sets and is closed under the operation of taking the image by a proper† real-analytic map.

Triangulations and stratifications exist for both Tthe semianalytic and subanalytic sets ([5], [3], [4]), but the proofs are non-trivial. Unfortunately, such sets are not closed under projection and sets with infinitely many connected components exist in both classes.

The class of *finitely subanalytic* sets is defined ([19]) as those subanalytic sets which remain subanalytic under an analytic mapping into the unit n-cube (or equivalently, those subanalytic sets in $\mathbb{R}^n$ which are also subanalytic in the extension of $\mathbb{R}^n$ to projective space $P^n(\mathbb{R})$). This set excludes the complete sine curve, but includes any bounded region of that curve. In contrast to the semianalytic and subanalytic sets, both the semialgebraic sets and the finitely subanalytic sets form an *O-minimal structure*:

O-minimal Structure: An *O-minimal structure* on $\mathbb{R}$ is a sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ such that for each n:

1. $\mathcal{A}_n$ is a boolean algebra of subsets of $\mathbb{R}^n$.
2. $(A \in \mathcal{A}_n) \Rightarrow (A \times \mathbb{R} \in \mathcal{A}_{n+1}$ and $\mathbb{R} \times A \in \mathcal{A}_{n+1})$.
3. $\{\langle x_1, \dots, x_n\rangle \in \mathbb{R}^n \mid x_i = x_j\} \in \mathcal{A}_n$ for $1 \le i < j \le n$.
4. $(A \in \mathcal{A}_{n+1}) \Rightarrow (\pi(A) \in \mathcal{A}_n)$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection onto the first n-coordinates.
5. $\{r\} \in \mathcal{A}_1$ for each $r \in \mathbb{R}$, and $\{(x, y) \in \mathbb{R}^2 \mid x < y\} \in \mathcal{A}_2$
6. The only sets in $\mathcal{A}_1$ are the finite unions of open (possibly unbounded) intervals and points.

These conditions are designed to capture closure under *simple logical constructions* (see below) but exclude collections which “blow up” to include sets with infinitely many components. The first four conditions ensure that the class is closed under construction by logical operations on defining predicates. Condition (1) covers the familiar boolean operations that correspond to intersection, union and complement; (2) is the model theoretic interpretation of universal quantification (∀); (3) covers equality constraints on variables (a generalisation of diagonal sets); (4) is the model theoretic interpretation of existential quantification (∃); and (5) ensures that all real points and the (<) predicate may be used in constructions. The first five conditions together ensure the inclusion of any set which can be described using a first order formula involving real points, equality, (<) and other definable sets. Such formulae admit constructions of frontier, closure, boundary and interior. The final condition (6) excludes all collections with infinitely many connected components because the other conditions ensure closure under rotation and orthogonal projection.

There are O-minimal structures which do not include the semialgebraic sets. However, their inclusion is guarenteed if we add the condition:

- 7. The graphs of functions (+) and (.) in $\mathbb{R}^2$ are in the collection.

The smallest O-minimal structure on $\mathbb{R}$ generated from (+) and (.) and bounded semianalytic sets is exactly the finitely subanalytic sets.

Generalisations of the semialgebraic sets enable geometric modelling systems to model trigonometric functions (e.g. to describe screwthreads and springs).

† A mapping is proper if the inverse image of compact sets are compact.

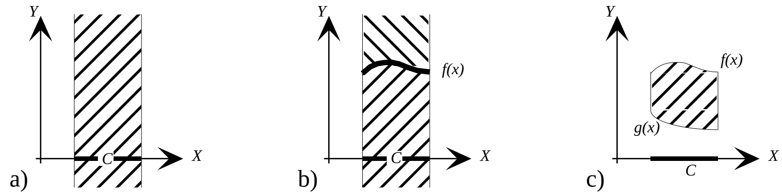


Figure 1: Icells in $\mathbb{R}^2$ defined over an Icell C in $\mathbb{R}^1$

Finitely subanalytic sets provide the trigonometric functions without sacrificing important properties because they are O-minimal structures, but there are even more inclusive O-minimal structures on $\mathbb{R}$ which include exponential functions [24]. These are the Pfaffian sets that have been shown to form O-minimal structures. They are particularly interesting for geometric modelling because they provide useful bounds on the complexity of constructions such as stratification.

Except where stated otherwise, this paper subsequently works with an arbitrary O-minimal structure on $\mathbb{R}$ that includes graphs of $(+)$ and $(.)$. This includes, as special cases, the semialgebraic sets and the finitely subanalytic sets.

A set is *definable* if it is a set in the O-minimal structure under consideration, and a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is definable if its graph $graph(f) \subseteq \mathbb{R}^{m+n}$ is definable. For the purposes of geometric modelling, it is important that a definable set can be decomposed into a finite collection of disjoint cells, each of which is a definable set of an especially simple nature. Such geometric modelling cells are discussed in subsection 3.1. For the moment, we use the definition of cell from [19] which we refer to as *Icell* (since they are inductively defined):

Icell: The set containing the single point of $\mathbb{R}^0$ is an Icell and an Icell in $\mathbb{R}^{n+1}$ is formed *over* an Icell $C \subseteq \mathbb{R}^n$ in one of the following ways:-

- $C \times \mathbb{R}$. This is the unbounded cylinder with cross section C (e.g. fig. 1(a) where $n = 1$).
- $\{\langle x, r\rangle \in C \times \mathbb{R} \mid r < f(x)\}, \{\langle x, r\rangle \in C \times \mathbb{R} \mid r > f(x)\}$, and $graph(f)$, where $f : C \rightarrow \mathbb{R}$ is a definable continuous function. These cells are regions bounded by the single function f and the graph of that function (e.g. fig. 1(b)).
- $\{\langle x, r\rangle \in C \times \mathbb{R} \mid g(x) < r < f(x)\}$, where $f, g : C \rightarrow \mathbb{R}$ are definable continuous functions such that $g(x) < f(x)$ for all $x \in C$. This cell fills the region between the bounding functions f and g (e.g. fig. 1(c)).

The Icells in $\mathbb{R}^1$ are single points $\{r\}$ or single intervals $(a \ldots b)$ where $a < b$, including the unbounded cases $(-\infty \ldots b)$, $(a \ldots +\infty)$ and $(-\infty \ldots +\infty) = \mathbb{R}$.

Decomposition: A *decomposition* of $\mathbb{R}^n$ is a partition of $\mathbb{R}^n$ into finitely many Icells that satisfy projection conditions defined by induction on $n \geq 0$ as follows:

- A decomposition of $\mathbb{R}^0$ is simply $\{\mathbb{R}^0\}$.
- A decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into Icells C such that the set of projections $\pi(C)$ is a decomposition of $\mathbb{R}^n$.

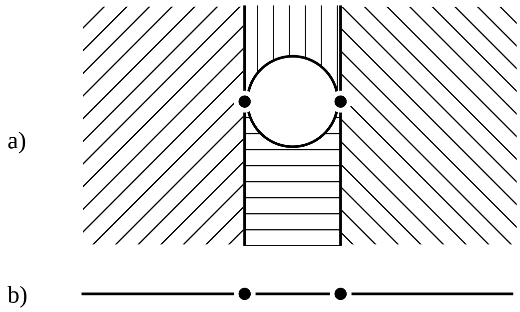


Figure 2: (a) A decomposition of $\mathbb{R}^2$ and (b) its projection on $\mathbb{R}^1$

A decomposition of $\mathbb{R}^1$ takes the form
$\{(-\infty \ldots a_1), (a_1 \ldots a_2), \ldots (a_k \ldots +\infty), \{a_1\}, \ldots \{a_k\}\}$, (for $a_1 < a_2 < \ldots a_k$). Figure 2 shows the simplest decomposition of $\mathbb{R}^2$ that includes a unit open disc centered at $\langle 2, 2\rangle$. The disc boundary must be subdivided as shown in order that its projection onto the x-axis ($\mathbb{R}^1$) is an open interval.

Theorem 1 *(Icell decomposition, [20] 2.11, p.52)*

- *Given definable sets $A_1, \ldots, A_k \subseteq \mathbb{R}^n$, there is a decomposition of $\mathbb{R}^n$ that partitions each of the sets $A_1, \ldots, A_k$.*
- *For each definable function $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$, there is a decomposition $\mathcal{D}$ of $\mathbb{R}^n$ that partitions A such that the restriction $f|C : C \to \mathbb{R}$ to each Icell $C \in \mathcal{D}$ with $C \subseteq A$ is continuous.*

The second part of this theorem states that the domain of a definable function (which is also definable by the projection property) can be partitioned by a decomposition into regions where the function is continuous. Both parts of the theorem may be proved simultaneously by induction on n.

3 Cellular geometric models and stratification

Objects in the most recent geometric modelling systems are structured as unions of disjoint cells. However, since the nature of such cells and the cell sets which constitute a valid object are ill-defined, it is not possible for the results of object creation and combination to be defined precisely. The restrictive nature of the previously introduced Icells and decompositions makes the decomposition theorem very powerful, but Icell decompositions are not suitable for representing the cellular objects of geometric modelling systems. This section introduces a more suitable general form of stratification (decomposition) that admits non-smooth cells.

3.1 Cells of Geometric Models

Consider objects composed of a finite number of connected cells that are (at least) topological submanifolds of $\mathbb{R}^n$ ($0 \leq n$):

Submanifold: A (topological or C^0) submanifold of $\mathbb{R}^n$ of dimension $m \geq 0$ is a set $M \subseteq \mathbb{R}^n$ such that each point in M has a neighbourhood in $\mathbb{R}^n$ whose intersection with M is homeomorphic to an m-ball.

This allows, for example, the surface of a cube to be a single cell. Djinn objects [9] have finitely many cells that are semianalytic sets and topological submanifolds of $\mathbb{R}^n$ ($0 \leq n \leq 3$). There is no constraint which prohibits objects with infinitely many connected components. For example the (semianalytic) intersection of the sets $\{\langle x, sin(\pi x)\rangle | \; x \in \mathbb{R}\}$ and $\{\langle x, 0\rangle | \; x \in \mathbb{R}\}$ in $\mathbb{R}^2$ is an infinite set of isolated points. Unfortunately, the inclusion of such sets leads to the generation of more and more complex sets that cause serious problems in geometric modelling systems (see, for example, the discussion in the introduction of [20]). The use of O-minimal structures rules out such problematic sets by the requirement that cells are connected and thus avoids the many further complications discussed in [10]).

Cells appropriate for geometric modelling are defined as follows:

Dcell: A *Dcell* is a definable set (in an O-minimal structure) that is a connected, C^0-submanifold of $\mathbb{R}^n$ for some $n \geq 0$.

Since each definable set can be decomposed into finitely many smooth Icells [20], the requirement that Dcells are definable also ensures that they are at least piecewise smooth.

A circle in $\mathbb{R}^2$ is an example of a Dcell that is not a single Icell. It can be decomposed into Icells, e.g. two semi-circles and their two common end points (fig. 2). The existence of such decompositions is important mathematically, but the additional structure is usually an unnecessary artifact for applications[‡]. Hence we use the more general Dcell for structuring geometric models.

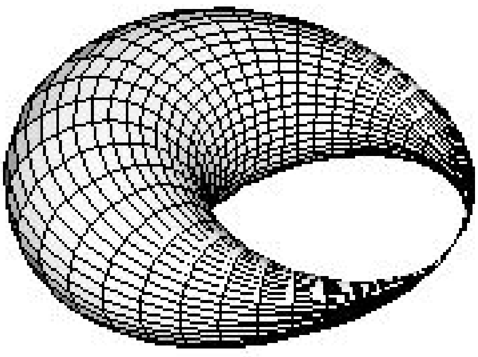
3.2 Stratification of Geometric Models

There are various definitions of stratification in the literature. Stratifications are disjoint unions of submanifolds that are usually smooth (C^∞) or at least C^r-continuous for some $r > 0$. In addition, they are usually required to be at least locally finite (i.e. to contain finite numbers of strata in each bounded region) if not finite, and the strata are usually (but not always) connected. Further properties are often imposed on stratifications, e.g. the frontier condition: any stratum that intersects the frontier of another stratum lies entirely in the latter's frontier[§].

Whitney stratifications [22] are at least at least C^r-continuous for some $r > 0$ and satisfy the Whitney condition: As a sequence of points in one stratum approaches a point in a frontier stratum, the limit of line segments from the points to that frontier point lies in the tangent space at that frontier point, and the tangent spaces of the points similarly converge on the tangent space at that frontier point. For stratifications of closed sets, the Whitney condition is equivalent to the requirement that for any two points in the same stratum, there are neighbourhoods of the points with a diffeomorphism between them that respects strata. This implies the frontier condition for stratifications of closed sets.

[‡] A similar situation arises in early BRep modellers which sometimes require edges that are artificial from the point of view of the application.

[§] The *frontier* of a set A is the set of points in the closure of A but not in A, i.e. $frn(A) =_{\mathbf{def}} clo(A) - A$. The frontier is usually different from the boundary $bnd(A) =_{\mathbf{def}} clo(A) - int(A)$, where $int(A)$ is the interior of A; boundary and frontier coincide only for open sets.



Strata :
$P_C = P_0 =$ pinch point
$L_1 =$ circle through P_C
$A_C =$ surface - P_C
$A_1 = A_0 =$ surface - L_1
$L_0 =$ circle $L_1 - P_C$

Stratifications :
$\mathcal{S}_C = \{P_C, A_C\}$
$\mathcal{S}_1 = \{L_1, A_1\}$ (non-topological)
$\mathcal{S}_0 = \{P_0, L_0, A_0\}$

Figure 3: Stratifications of a cyclide with one pinch point

Stratifications of definable sets into Dcells, cannot exploit the Whitney condition because the strata may not be C^1-continuous. Instead, we restrict attention to closed sets and impose a weaker topological condition that also guarantees the frontier condition.

Topological Stratification: A *topological stratification* $\mathcal{S}$ of a closed definable set $A \subseteq \mathbb{R}^n$ is a finite partition of A into non-empty Dcells (strata) that satisfy the *topological uniformity condition*: any two points in the same stratum have neighbourhoods with a homeomorphism between them that respects strata.

In the context of an 0-minimal structure, definable sets are guaranteed to contain finitely many topological types. The topological uniformity condition for a Dcell stratification of a closed set implies that each stratum is topologically uniform with respect to each of the other strata:

Topological uniformity: If X, Y are disjoint definable sets, X is *topologically uniform with respect to* Y if all points in X have homeomorphic neighbourhoods in $X \cup Y$.

It is easy to verify that the frontier condition is a consequence of topological uniformity: if C and C' are strata and $C \cap \mathit{frn}(C') \neq \emptyset$, $C \subseteq \mathit{frn}(C')$.

For an example of the topological uniformity condition, consider a cyclide surface with a single pinch point (fig. 3). The surface is non-manifold (due to the pinch point), but there is an obvious topological stratification $\mathcal{S}_C$ into the pinch point (a zero dimensional stratum) and the rest of the surface (a 2 dimensional stratum). It is also possible to partition the surface into $\mathcal{S}_1$ consisting of a circle passing through the pinch point (a 1 dimensional stratum) and the rest of the surface. These two manifolds satisfy the frontier condition, but $\mathcal{S}_1$ is not a topological stratification because the pinch point has a topological relationship to the surface stratum that is different from that of all other points on the circle stratum. Splitting the circle into 2 strata (the pinch point and the rest) produces a valid topological stratification $\mathcal{S}_0$.

Stratifications are important in geometric modelling for navigating cellular structure via frontier relations. In O-minimal structures, frontiers are always of lower dimension but this is not necessarily the case for arbitrary subsets of $\mathbb{R}^n$ (e.g. the topologists curve [8]).

3.3 Orders for stratifications

The set of all possible stratifications of a closed set inherit the natural order of partition *refinement*. To be precise, $P \leq P'$ if P and P' are partitions of the same set and P is refined by P', i.e. each set in the collection P' is a subset of a set in the collection P. This provides a partial order where lower in the order means fewer (but larger) sets in the partition. In the cyclide example discussed in the previous section, $\mathcal{S}_C < \mathcal{S}_0$. Unfortunately, a stratification that is least (below all others) with respect to this order does not necessarily exist. For example, the (non-topological) stratification $\mathcal{S}_1$ of the cyclide into a circle and the rest of the surface is neither below nor above the stratification $\mathcal{S}_C$; there is no least stratification.

In the literature (e.g. [6]), a related but different filtration order is introduced to resolve this problem:

Filtration Order: $\mathcal{S} \sqsubseteq \mathcal{S}'$ if there exists $j \geq 0$ such that $\langle \mathcal{S} \rangle_i = \langle \mathcal{S}' \rangle_i$ for all $i > j$ and $\langle \mathcal{S}' \rangle_j \subset \langle \mathcal{S} \rangle_j$, where $\langle \mathcal{S} \rangle_i$ is the union of all strata of dimension at most i in stratification $\mathcal{S}$. (The sequence $\langle \mathcal{S} \rangle_i, i \geq 0$ is called a filtration of $\mathcal{S}$.)

It is easy to verify that $\mathcal{S} \leq \mathcal{S}' \implies \mathcal{S} \sqsubseteq \mathcal{S}'$, but the converse is not true. For Whitney stratifications, the *canonical* Whitney stratification is defined to be the least Whitney stratification (when it exists) with respect to the filtration order.

Stratifications discussed in the following sections are topological stratifications except where explicitly stated otherwise. "Least" stratification means "least with repect to the refinement order" whilst the term *canonical* is used for "least with repect to the filtration order".

4 Least stratifications

This section contains the main result of the paper, i.e. the existence of least topological stratifications. It uses a generalised refinement which allows collections of point-sets which are not necessarily partitions.

Compatible: A subset X of a set A is *compatible* with a collection of subsets $\mathcal{Y} = \{Y_i \subseteq A\}_{i \in I}$ if, for each $i \in I$, either $X \cap Y_i = \emptyset$ or $X \subseteq Y_i$. A collection of subsets $\mathcal{X}$ is compatible with a collection $\mathcal{Y}$ if each $X \in \mathcal{X}$ is compatible with $\mathcal{Y}$

In particular, a partition of a set A is compatible with subsets $\{Y_i \subseteq A\}_{i \in I}$ if and only if it contains a partition of each Y_i. If $\mathcal{X}$ and $\mathcal{Y}$ are both partitions of A, then $\mathcal{X}$ is compatible with $\mathcal{Y}$ if and only if $\mathcal{X}$ is a refinement of $\mathcal{Y}$ ($\mathcal{Y} \leq \mathcal{X}$) .

We subsequently prove that there exists a least topological stratification of a space $\mathbb{R}^n$ that is compatible with a given finite collection of definable sets. That is, if $\mathcal{D}$ is a finite collection of definable subsets of $\mathbb{R}^n$, there is a least topological stratification of $\mathbb{R}^n$ that is compatible with $\mathcal{D}$. This result is subsequently used to define the least upper bound (or join) of two stratifications that is the basis for definitions of "intersection" and "union" of cellular object models. First, we describe a construction similar to Mather's construction of canonical Whitney stratifications, but we replace the Whitney regularity condition with conditions that ensure a topological stratification is created.

The following two Lemmas are needed to ensure uniqueness of the construction.

Lemma 1 *If $A \subseteq \mathbb{R}^n$ is a definable set and $X, Y \subseteq A$ are definable non-empty connected sub-manifolds of A that are open in A,* ***either*** *$X \cup Y$ is not connected (equivalently $clo(X) \cap Y = clo(Y) \cap X = \emptyset$),* ***or*** *$X \cup Y$ is a non-empty connected sub-manifold that is open in A*

Figure 4 shows some examples in $\mathbb{R}^2$ that illustrate the need for the relatively open condition. In cases (a) and (b), X and Y are relatively open in $A = clo(X \cup Y)$ but $X \cup Y$ is connected only in (b). In cases (c) and (d), X is not relatively open in A and $X \cup Y$ is connected but not manifold.

	$A = clo(X \cup Y)$	
	X	Y
(a)	$\{(x, 0) \mid x < 0\}$	$\{(x, 0) \mid x > 0\}$
(b)	$\{(x, 0) \mid x < 1\}$	$\{(x, 0) \mid x > 0\}$
(c)	$\{(x, 0) \mid x \in \mathbb{R}\}$	$\{(0, y) \mid y > 0\}$
(d)	$\{(x, 0) \mid x \in \mathbb{R}\}$	$\{(0, y) \mid y \in \mathbb{R}\}$

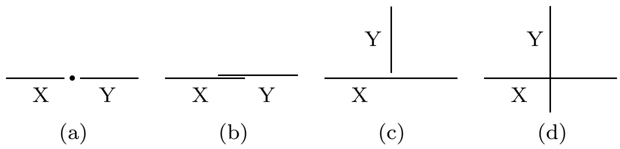


Figure 4: Unions of connected manifolds X and Y

Proof: Suppose X, Y are definable non-empty connected sub-manifolds that are open in A. Clearly, $X \cup Y$ is non-empty and open in A. Definable sets that are open in a definable set A have the same dimension as A; let m be the dimension of X, Y, $X \cup Y$ and A. Suppose that $X \cup Y$ is connected but not a sub-manifold. A contradiction is obtained by considering all possible cases. If $x \in X \cup Y$ is a point that does not have an m-manifold neighbourhood in $X \cup Y$, one of the following four cases must hold:

1. $x \in X - clo(Y)$: Since x does not have an m-manifold neighbourhood in X, this contradicts X being manifold.
2. $x \in Y - clo(X)$: Similar contradiction (by symmetry).
3. $x \in X \cap clo(Y)$: If any neighbourhood of x in $X \cup Y$ did not contain points in $Y - X$, it would also be a neighbourhood of x in X, contradicting $X \cup Y$ non-manifold at x. Therefore, since every neighbourhood of x in $X \cup Y$ contains points in $Y - X$, X is not open in $X \cup Y$ and hence X is not open in A, a contradiction.
4. $x \in Y \cap clo(X)$: Similar contradiction (by symmetry).

Thus we have a contradiction in each case, concluding the proof. □

Topologically uniform collection: For a finite collection $\mathcal{B}$ of disjoint Dcells in $\mathbb{R}^n$, $\mathcal{B}$ is a *topologically uniform collection* if each $B \in \mathcal{B}$ is topologically uniform with respect to each of the other cells in $\mathcal{B}$.

Similarly, for a Dcell C disjoint from each $B \in \mathcal{B}$ we say that C is topologically uniform with respect to $\mathcal{B}$ if it is topologically uniform with respect to each $B \in \mathcal{B}$.

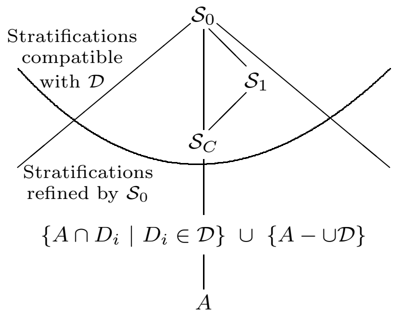


Figure 5: Some stratifications of A and a partition of A compatible with $\mathcal{D}$

Lemma 2 *Let $\mathcal{B}$ be a topologically uniform collection of Dcells of dimension strictly greater than m and let X, Y be Dcells of dimension m which are topologically uniform with respect to $\mathcal{B}$. If $X \cap Y \neq \emptyset$ and $X \cup Y$ is an m-dimensional Dcell, $X \cup Y$ is topologically uniform with respect to $\mathcal{B}$*

Proof: Since the intersection of X and Y is non-empty, points in the intersection have the same topological type in $X \cup Y \cup B$ as all other points in X and Y have in $X \cup Y \cup B$, for each $B \in \mathcal{B}$. □

Lemmas 1 and 2 are used to prove the following special case of the existence of canonical stratifications that are refined by a given stratification.

Lemma 3 *If a closed definable set $A \subseteq \mathbb{R}^n$ has a topological stratification $\mathcal{S}_0$ that is compatible with a finite collection of definable sets $\mathcal{D}$ in $\mathbb{R}^n$, there exists a canonical (least in the filtration order) topological stratification $\mathcal{S}_C$ of A which is compatible with $\mathcal{D}$ and refined by $\mathcal{S}_0$.*

This situation is illustrated by figure 5. The area above the parabola and below the open triangle from $\mathcal{S}_0$ contains all stratifications of A which are both refined by $\mathcal{S}_0$ and compatible with $\mathcal{D}$. The partition of A into $\{A \cap D_i | D_i \in \mathcal{D}\} \cup \{A - \bigcup \mathcal{D}\}$ is compatible with $\mathcal{D}$ but it is not necessarily a stratification.

The subsequent proof uses induction on the dimension of A. The construction of $\mathcal{S}_C$ involves the amalgamation of cells in $\mathcal{S}_0$ starting with the highest dimension whilst maintaining the required properties (compatibility with $\mathcal{D}$, etc.). At lower dimensions the construction of cells maintains topological uniformity with higher dimensional cells already constructed. This requires proof of a stronger result than that of the lemma:

Let $\mathcal{B}$ be a topologically uniform collection of cells disjoint from A and of strictly higher dimension than A such that each Dcell in $\mathcal{S}_0$ is also topologically uniform with respect to $\mathcal{B}$. We prove (for any dimension m and A a closed definable set with that dimension) that *there exists a stratification $\mathcal{S}_C$ (as in the lemma) such that the cells of $\mathcal{S}_C$ are also topologically uniform with respect to $\mathcal{B}$.* The lemma is thus a special case where $\mathcal{B}$ is an empty collection. In the inductive proof we add cells to $\mathcal{B}$ to ensure an appropriate stratification with the cells of lower dimension:

Proof: If A is empty the result is trivial (an empty set is usually considered to have a dimension of -1). If A has dimension 0, it consists of a finite set of isolated points for which the result is again trivial. Now suppose $dimA = m > 0$ and that the inductive hypothesis holds for closed definable sets of lower dimension. Let stratification $\mathcal{S}_0 = \{C_1, \ldots, C_k\}$. For convenience, C_I is used as an abbreviation for an amalgamation of cells $\bigcup_{i \in I} C_i$, where $I \subseteq \{1, \ldots, k\}$.

Consider cell C_i such that $dimC_i = dimA = m$ and define $[i]$ as a maximal subset of $\{1, \ldots, k\}$ such that

1. $i \in [i]$ (i.e. $C_i \subseteq C_{[i]}$),
2. $C_{[i]}$ is a connected sub-manifold of dimension m (if [i] is not a singleton, some of its components have lower dimension),
3. $C_{[i]}$ is open in A,
4. $C_{[i]}$ is topologically uniform with respect to $\mathcal{B}$, and
5. $C_{[i]}$ is compatible with $\mathcal{D}$.

Such a maximal index set $[i]$ is uniquely determined because two maximal sets have at least i in common, the corresponding manifolds overlap on C_i and their union also satisfies the conditions (condition 2 and 3 follow by Lemma 1, condition 4 follows by Lemma 2 and condition 5 is obvious). Since maximality of $[i]$ excludes a larger set that satisfies the conditions, the two sets are equal. Similarly (by maximality and the lemmas), any two maximal sets $[i]$ and $[j]$ are either disjoint or equal. Thus if $I_1, \ldots, I_r$ are the disjoint subsets of $\{1, \ldots, k\}$ that are elements of $\{[i] | i \in 1, \ldots, k \wedge dimC_i = m\}$, the cells of $\mathcal{S}_C$ of dimension m are $C_{I_1}, \ldots, C_{I_r}$.

The remaining cells from $\mathcal{S}_0$ that cannot be amalgamated to form an m-submanifold are $\{C_j\}_{j \in J}$, where $J = \{1, \ldots, k\} - I_1 - \cdots - I_r$. These cells cover $A' =_{\mathbf{def}} A - C_{I_1} - \cdots - C_{I_r} = C_J$ and A' has the following properties:

- A' is closed because the removed $C_{I_1} \ldots C_{I_r}$ are open in A and A is closed.
- $dimA' < dimA$ because A' is covered by cells of dimension less than m.
- $\{C_j\}_{j \in J}$ is a stratification of A' which is compatible with $\mathcal{D}$ and has cells topologically uniform with respect to $\mathcal{B} \cup \{C_{I_1}, \ldots, C_{I_r}\}$ because the original stratification had cells topologically uniform with respect to $\mathcal{B}$ and each of the Dcells $C_{I_1}, \ldots, C_{I_r}$ is open in A and disjoint from the remaining cells.

Therefore, by the induction hypothesis, there exists a canonical stratification $\mathcal{S}'$ of A' which is refined by the stratification $\{C_j\}_{j \in J}$, compatible with $\mathcal{D}$ and contains cells which are topologically uniform with respect to $\mathcal{B} \cup \{C_{I_1}, \ldots, C_{I_r}\}$. The required canonical stratification of A is $\mathcal{S}_C =_{\mathbf{def}} \mathcal{S}' \cup \{C_{I_1}, \ldots, C_{I_r}\}$.

By maintaining condition 4 and uniting cells of higher dimension with collection $\mathcal{B}$, all new cells are topologically uniform with respect to cells of strictly higher dimension. Condition 3 ensures that new cells are topologically uniform with respect to cells of the same dimension. It follows that $\mathcal{S}_C$ is a topological stratification of A. The construction ensures that $\mathcal{S}_C$ is refined by $\mathcal{S}_0$ and is compatible with $\mathcal{D}$. It is a least stratification in the filtration ordering due to the construction of maximal cells from the highest dimension downward. □

Lemma 4 *If $\mathcal{S}_C$ is the canonical (least in the filtration order) stratification refined by a stratification $\mathcal{S}_0$ and compatible with a finite collection of definable sets $\mathcal{D}$, then it is also the least such stratification (with respect to the refinement order).*

Proof: The proof is by contradiction. Suppose $\mathcal{S}_1$ is another stratification refined by $\mathcal{S}_0$ and compatible with $\mathcal{D}$ but $\mathcal{S}_1$ is not a refinement of $\mathcal{S}_C$. We choose a cell $L_1 \in \mathcal{S}_1$ (of maximal dimension) such that L_1 does not lie entirely within a single cell of $\mathcal{S}_C$, and then use topological uniformity and the canonical construction of $\mathcal{S}_C$ to derive a contradiction.

An example to keep in mind is the cyclide shown in figure 3. The canonical stratification $\mathcal{S}_C$ separates the pinch point from the rest of the surface. An incompatible stratification (such as $\mathcal{S}_1$) contains a cell that incorporates the pinch point and other points in the surface ($L_1 \in \mathcal{S}_1$) and therefore cannot be topologically uniform.

Consider the highest dimension (1 for the cyclide) for which there is a cell $L_1 \in \mathcal{S}_1$ that is not contained in a single cell of $\mathcal{S}_C$. Let A_C be the highest dimension cell of $\mathcal{S}_C$ that intersects L_1. Note that $dim L_1 \leq dim A_C$.

Suppose that $dim L_1 = dim A_C$. Let $\mathcal{T}_1$ be the set of cells in $\mathcal{S}_1$ that have L_1 in their frontier. These cells are embedded in a set of cells $\mathcal{T}_C$ in $\mathcal{S}_C$ that have A_C in their frontier. The choice of L_1 ensures that each cell in $\mathcal{S}_1$ of higher dimension than that of L_1 is embedded in a single corresponding cell in $\mathcal{S}_C$. Topological uniformity of $\mathcal{S}_1$ means that points in L_1 are topologically uniform with respect to cells in $\mathcal{T}_1$ and therefeore to cells in $\mathcal{T}_C$. But this contradicts the fact that L_1 does not form a subset of A_C in the canonical construction of $\mathcal{S}_C$.

The same contradiction is obtained when $dim L_1 < dim A_C$ but A_C is included in the set of cells $\mathcal{T}_C$.

Therefore $\mathcal{S}_1$ is a refinement of $\mathcal{S}_C$ and $\mathcal{S}_C$ is refined by all stratifications that are compatible with $\mathcal{D}$ and refined by S_0. □

The existence of a topological stratification of any definable set is guaranteed:

Lemma 5 *If A is a closed definable subset of $\mathbb{R}^n$ and $\mathcal{D}$ is a finite collection of definable subsets of A, there exists a topological stratification of A into (finitely many) Dcells that is compatible with $\mathcal{D}$.*

Proof: It is possible to refine an Icell decomposition of A to ensure that the cells are C^1-continuous and that the Whitney condition is met for neighbouring cells. Since the Whitney condition implies topological uniformity (for stratifications of closed sets) and since Icells are also Dcells, this establishes the existence of a Dcell topological stratification of A that is compatible with $\mathcal{D}$. □

Theorem 2 *If $\mathcal{A}$ is a finite collection of definable subsets of $\mathbb{R}^n$, there is a least stratification of $\mathbb{R}^n$ that is compatible with $\mathcal{A}$.*

Proof: By lemma 5 there is a stratification $\mathcal{S}_0$ of $\mathbb{R}^n$ that is compatible with $\mathcal{D}$, and by lemmas 3 and 4 there is a least stratification $\mathcal{S}$ refined by $\mathcal{S}_0$ that is compatible with $\mathcal{D}$ (fig. 6). The remainder of the proof shows that $\mathcal{S}$ is the least of all stratifications of $\mathbb{R}^n$ that are compatible with $\mathcal{D}$, i.e. that any stratification $\mathcal{S}_1$ of $\mathbb{R}^n$ which is compatible with $\mathcal{D}$ (not necessarily refined by S_0) satisfies $\mathcal{S} \leq \mathcal{S}_1$.

By lemma 5 there is a stratification $\mathcal{S}_2$ of $\mathbb{R}^n$ that is compatible with $\mathcal{D}$ and also compatible with the sets in $\mathcal{S}_0$ and $\mathcal{S}_1$ (fig. 6). Let $\mathcal{S}_3$ be the least stratification of $\mathbb{R}^n$ refined by $\mathcal{S}_2$ which is compatible with $\mathcal{D}$ ($\mathcal{S}_3$ exists by lemmas 3 and 4). Since $\mathcal{S}_2$ refines both $\mathcal{S}_0$ and $\mathcal{S}_1$, $\mathcal{S}_0 \leq \mathcal{S}_2$ and $\mathcal{S}_1 \leq \mathcal{S}_2$. Therefore, $\mathcal{S}_3 \leq \mathcal{S}_0$ and $\mathcal{S}_3 \leq \mathcal{S}_1$ because $\mathcal{S}_3$ is the least stratification refined by $\mathcal{S}_2$ that is compatible with $\mathcal{D}$. Since $\mathcal{S}_3 \leq \mathcal{S}_0$ and $\mathcal{S}$ is the least stratification refined by $\mathcal{S}_0$ that is compatible with $\mathcal{D}$, $\mathcal{S} \leq \mathcal{S}_3$. Therefore, $\mathcal{S} \leq \mathcal{S}_3 \leq \mathcal{S}_1$, and $\mathcal{S} \leq \mathcal{S}_1$ as required. □

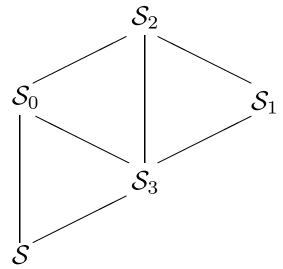


Figure 6: Ordering of stratifications in theorem 2

Leaststrat: Define $leaststrat_n\mathcal{D}$ to be the least topological stratification of $\mathbb{R}^n$ (with respect to the refinement order) that is compatible with a finite collection of definable sets $\mathcal{D}$.

Theorem 2 can be used to show that least upper bounds exist for pairs of stratifications that are compatible with another collection. A least upper bound $\mathcal{S}_0 \sqcup \mathcal{S}_1$ of $\mathcal{S}_0$ and $\mathcal{S}_1$ is a stratification that is least (with respect to the refinement order) of all the stratifications that are refinements of both $\mathcal{S}_0$ and $\mathcal{S}_1$; $\mathcal{S}_0 \sqcup_{\mathcal{D}} \mathcal{S}_1$ is such a bound amongst stratifications that are compatible with $\mathcal{D}$:

Corollary 1 *If $\mathcal{S}_0$ and $\mathcal{S}_1$ are two stratifications of $\mathbb{R}^n$ that are compatible with a finite collection of definable sets $\mathcal{D}$, there is a least upper bound of $\mathcal{S}_0$ and $\mathcal{S}_1$ amongst all stratifications of $\mathbb{R}^n$ that are compatible with $\mathcal{D}$.*

Proof: An upper bound of $\mathcal{S}_0$ and $\mathcal{S}_1$ is a stratification that refines both $\mathcal{S}_0$ and $\mathcal{S}_1$. Since a refinement of a stratification is compatible with that stratification, $leaststrat_n(\mathcal{D} \cup \mathcal{S}_0 \cup \mathcal{S}_1)$ is the required least upper bound. □

A cell in $\mathcal{S}_0 \sqcup \mathcal{S}_1$ is not necessarily a simple intersection of a cell from each argument because the intersection of two Dcells is not always a single Dcell, e.g. two manifold curves could cross at a point and have a curve segment in common. Separate stratification of each such intersection produces Dcells, but the collection of all such stratifications may not satisfy the frontier condition. However, valid stratifications always can be produced by further refinement. This is clearly illustrated by a $3D$ example in [8].

5 Geometric models and operations

5.1 Objects

In previous formalisations [9], a Djinn object was defined as a stratification of a semianalytic closed set into Djinn cells that are (orientable but not necessarily connected) C^0-submanifolds of $\mathbb{R}^n$ $(0 \leq n \leq 3)$. In addition to the frontier condition, these stratifications satisfy constraints on cells with more than one connected component and constraints on the relative orientation between a cell and its frontier. However, object models and their "set-like" combination operations are greatly simplified if their cells (i.e. strata) are constrained to be connected (unlike Djinn) but the potentially infinite numbers of connected components correspond to infinite stratifications. In contrast, definable sets (i.e. sets in an

O-minimal structure on $\mathbb{R}^n$) always have finitely many connected components, and this is central to many of the simplifications subsequently discussed.

Since the complement and closure of a definable set is also definable, a stratification of a closed definable set is converted into a stratification of space by including the (necessarily open) external components as additional strata. A Djinn* object [8] is a stratification of space together with an indication of which strata (cells) represent the "inside" of the object being modelled. Formally, generalising Djinn* from subanalytic to O-minimal:

Object: A Djinn* *object* O is a pair $\langle \mathcal{S}, P \rangle$, where $\mathcal{S}$ is a (Dcell) stratification of $\mathbb{R}^n$ for some $n \geq 0$ and $P \subseteq \mathbb{R}^n$ is a definable set (the point-set of O) such that $\mathcal{S}$ is compatible with P.

Compatibility ensures that the point-set P is a union of cells of the object (i.e. strata of $\mathcal{S}$):

$$\forall C \in \mathcal{S},\ \forall x, y \in C,\ (x \in P) \iff (y \in P)$$

The spatial dimension of O, i.e. the unique $n \geq 0$ such that $\bigcup \mathcal{S} = \mathbb{R}^n$ where $O = \langle \mathcal{S}, P \rangle$, is denoted *sdim* O.

The Djinn insistence on oriented cells prevents the definition of deterministic "set-like" combination operations because many result cell orientations must be arbitrarily chosen ([10]). This led to Djinn* cells (Dcells) which are orientable but not necessarily oriented. However, since many applications require oriented cells, Djinn* provides a mechanism to assign a cell orientation and that orientation is preserved through object combinations whenever possible. Hence, Djinn* "set-like" combinations depend on the following:

Conjecture 1 *If $\mathcal{S}_0$ and $\mathcal{S}_1$ are topological stratifications of $\mathbb{R}^n$ into orientable Dcells, $\mathcal{S}_0 \sqcup \mathcal{S}_1$ has orientable strata.*

This conjecture is easily proved up to dimension 3 ¶: Each cell in $\mathcal{S}_0 \sqcup \mathcal{S}_1$ is a subset of a cell in $\mathcal{S}_0$ and a cell in $\mathcal{S}_1$. Since a subset of an orientable cell of the same dimension must be orientable, only the intersection of two strictly higher dimensional cells can give rise to a non-orientable cell. In $\mathbb{R}^n$ with $n \leq 3$, only 2-cells can be non-orientable, and the intersection of 3-cells can only produce 3-cells (which are open sets in $\mathbb{R}^3$).

5.2 Set-Operations on Objects

The most primitive "set-like" object combinations in Djinn are designed to preserve the cellular structure from each operand. Separate operations for altering structure before or after combination provide the flexibility to preserve the structure of only one or neither operand. Similar primitive combination operations for Djinn* objects are defined in [8] using Z-schema but there is a more concise definition in terms of least upper bounds ($\sqcup$):

Union of objects:
$\langle \mathcal{S}_1, P_1 \rangle \cup^{\#} \langle \mathcal{S}_2, P_2 \rangle \ =_{\mathbf{def}}\ \langle \mathcal{S}_1 \sqcup \mathcal{S}_2, P_1 \cup P_2 \rangle$

Intersection of objects:
$\langle \mathcal{S}_1, P_1 \rangle \cap^{\#} \langle \mathcal{S}_2, P_2 \rangle \ =_{\mathbf{def}}\ \langle \mathcal{S}_1 \sqcup \mathcal{S}_2, P_1 \cap P_2 \rangle$

Complement of an object:
$\neg \langle \mathcal{S}, P \rangle \ =_{\mathbf{def}}\ \langle \mathcal{S}, (\bigcup \mathcal{S}) - P \rangle$.

¶ Since we do not have a proof for higher dimensional cases, the need for checks for orientability cannot be ruled out in dimensions greater than 3.

Since each stratum of an intersection or union is part (or all) of the intersection of a stratum from each operand, the resulting stratification is compatible with the resulting point-set. Compatibility is trivially true for object complements.

The rationale for intersection and union operators which "preserve" the operands' cellular structures is discussed in [10]. The set operations defined above constitute a boolean algebra of point-sets in a fixed spatial dimension (i.e. of objects modulo cellular structure). This contrasts with Djinn [10] where the restriction to objects with closed point-sets excludes this property. There is a "least structured" representative of each equivalence class of objects with the same point-set that can be constructed using the departitioning operator $\flat$ of the next section. In particular, the least representative (in the refinement ordering) of the class of empty objects for each spatial dimension n is $\bot_n =_{\mathbf{def}} \langle\{\mathbb{R}^n\}, \emptyset\rangle$.

5.3 Partitioning and Departitioning

Departitioning and partitioning operations cahnge an object's stratification whilst maintaining its point-set. One extreme case of departitioning is to remove all non-essential structure (i.e. structure that is not essential for object validity):

Departition: $\flat\langle\mathcal{S}, P\rangle \;=_{\mathbf{def}}\; \langle \mathit{leaststrat}_n\{P\}, P\rangle,$ where $n = \mathit{sdim}\ \langle\mathcal{S}, P\rangle$.

The result is a valid Djinn* object **only** if the resulting strata are orientable, which is not necessarily the case (e.g. when P is a moebius strip in $\mathbb{R}^3$). Stratifications of $\mathbb{R}^3$ into orientable strata that are compatible with a moebius strip P exist, but there is no least stratification among them. Thus, the above departitioning operation is a partial function on objects (i.e. it is not applicable if the resulting cells are not orientable). Outside the domain of this partial operation, finer (stepwise) departitioning is necessary [12].

Geometric modelling systems must also support the subdivision of cells into C^r-continuous components. The existence of space decompositions into smooth (C^∞) Icells (and C^r Icells, $r > 0$) is established in [21] for O-minimal structures in which differential continuity can be defined. Decompositions into analytic (C^ω) cells also exist for currently known O-minimal structures that extend a real closed field. By constraining Dcells to be C^r ($r > 0$), C^∞ or C^ω, we can generalise the construction of least stratifications to meet this additional criterion. For $r > 0$, the further Whitney condition can be imposed on stratifications to ensure that each point in a stratum has a neighbourhood which is diffeomorphic with any strata-respecting diffeomorphism. This result was shown for the semi-algebraic sets by Whitney [23]. Writing $\mathit{leaststrat}^r_n\mathcal{A}$ for the least Whitney stratification of $\mathbb{R}^n$ into C^r Dcells ($r > 0$, $r = \infty$ or $r = \omega$) that is compatible with a finite collection of definable sets $\mathcal{A}$, the function to subdivide cells of a Djinn* object into C^r sub-manifolds is defined as follows:

Partition: $\sharp^r\langle\mathcal{S}, P\rangle \;=_{\mathbf{def}}\; \langle \mathit{leaststrat}^r_n\mathcal{S}, P\rangle,$ where $n = \mathit{sdim}\ \langle\mathcal{S}, P\rangle$.

6 Conclusions and Related Work

This paper provides a mathematical framework and proofs of constructions which support a formalism for the objects and primitive "set-like" combination operators of the Djinn* geometric modelling kernel API [8] generalised to O-minimal structures. It proposes O-minimal structures as an appropriate framework in which to prove generic results concerning stratification and cellular structure. In particular, it suggests that extensions to the repertoire of geometric modelling primitives beyond the semialgebraic sets must proceed within such a framework in order to avoid the creation of geometrically "untame" sets within a system closed under the basic operations. Such a framework should be useful for defining alternative representation independent interfaces for geometric modelling kernels that support cellular objects of mixed dimension.

New stratification results are proved using Dcells introduced to represent the geometric structure of physical objects. These cells are distinct from the inductively defined Icells used by [19] to decompose definable sets. In particular, this paper proves that the least topological stratification in the refinement order is the same as the canonical topological stratification (least in the filtration order). Thus, multiple cell amalgamation operations lead to the same stratification when no more amalgamation is possible. Without the proof, geometric modelling systems would need to implement a complicated time consuming check on the filtration order after each tentative amalgamation in order to prevent a stratification from which the canonical stratification could not be reached. The importance of least stratification existence is explicit in our definitions of departition and partition. It is very important also because amalgamation occurs as part of "set-like" combination algorithms. This is evident from our definitions of object "union" and "intersection" in terms of the least upper bound of two stratifications that relies on the existence of least stratifications.

Cellular objects and dimension-deficient point-sets were not necessary in the more restricted domain of the early work on geometric modelling foundations (e.g. [14, 13]). Operations on cellular objects are found in [15, 1, 16, 7] but none of this work provides a complete formal foundation with proofs and there is a lack of a clear link between cells and semialgebraic or semianalytic-sets. Space decompositions for semi-algebraic sets are used in [18] to analyse problems of representation conversion (e.g. b-rep to CSG geometric model conversion). That paper discusses relationships between the algebras of operations such as "set-like" combinations, closure and connected-component and uses those operations in decompositions of space generated from a finite set of polynomials. In contrast, this paper deals with space decompositions using the far more general class of O-minimal structures.

The relevance of canonical Whitney stratifications is recognised in [18]. Although we have chosen to use Icells as the atomic building blocks for stratification into Dcells, variations that enforce smoothness and/or sign invariance of "atomic" sets (like Shapiro) could be used instead. However, since departitioning can form strata that are neither smooth nor sign-invariant, these conditions have not been used herein.

Some of the early theoretical papers on solid modelling used semianalytic sets as a basis (e.g. the r-sets of [14]), but discussions were primarily limited to theory in common with semialgebraic sets [17]. This is reflected by the restriction of most solid modelling implementations to semialgebraic sets. Although semianalytic sets are addressed in [17], problems of closure under operations, infinities of connected components and other undesirable sets have not been formally addressed in the literature. Recent extensions to the ACIS kernel admit the use of arbitrary analytic functions but this exposes that system to these difficulties even with simple well-behaved functions. These issues are addressed in [8] where the use of finitely sub-analytic sets is proposed as a solution. This paper has provided a more formal treatment and shown that the more general class of O-minimal structures has all the necessary properties identified in [8].

References

[1] F. Arbab. Set models and boolean operations for solids and assemblies. *IEEE Computer Graphics & Applications*, 10(11):76–86, November 1990.

[2] C. Armstrong, A. Bowyer, S. Cameron, J. Corney, G. Jared, R. Martin, A. Middleditch, M. Sabin, and J. Salmon. *Djinn: a geometric interface for solid modelling.* Information Geometers Ltd., Winchester, UK, 2000.

[3] A. Gabrielov. Projections of semi-analytic sets. *Funct. Anal. Appl.*, 2:282–291, 1968.

[4] H. Hironaka. Subanalytic sets. In Y. Kusunoki, S. Mizohata, M. Nagata, H. Toda, M. Yamaguti, and H. Yoshizawa, editors, *Number Theory, Algebraic Geometry and Commutative Algebra*, pages 453–493. Kinokuniya Book-Store Co., Ltd., Tokyo, Japan, 1973.

[5] S. Lojasiewicz. Triangulation of semi-analytic sets. *Annali Della Scuola Normale de Pisa*, 18:449–474, 1964.

[6] J. Mather. Notes on topological stability. Lecture Notes, Harvard University, 1970.

[7] A. Middleditch. Cellular models of mixed dimension. Technical Report BRU/CAE/92:3, Department of Computer Science, Brunel University, West London, 1992.

[8] A. Middleditch and C. Reade. Improvements for the Djinn API to a geometric modelling kernel. *International Journal of Shape Modeling*, 6(1):79–103, June 2000.

[9] A. Middleditch, C. Reade, and A. Gomes. Rationale and formal definitions for the objects of the Djinn API to a geometric modelling kernel. Technical Report BRU/CAE/98:2, Brunel University, September 1998.

[10] A. Middleditch, C. Reade, and A. Gomes. Set-combinations of the mixed dimension cellular objects of the Djinn API. *Computer Aided Design*, 31:683–694, 1999.

[11] A. Middleditch, C. Reade, and A. J. Gomes. Point-sets and cell structures relevant to computer aided design. *International Journal of Shape Modeling*, 6(2):175–205, 2000.

[12] C. Reade and A. Middleditch. Partitioning and departitioning mixed dimension cellular objects of the djinn api. *(Submitted for publication)*, 2001.

[13] A. Requicha. Mathematical models of rigid solid objects. Production Automation Project Tech. Memo 28, University of Rochester, 1977.

[14] A. Requicha. Representations of rigid solids: Theory, methods, and systems. *ACM Computing Surveys*, 12(4):437–464, December 1980.

[15] J. Rossignac and M. O'Connor. SGC: A dimension-independent model for pointsets with internal structures and incomplete boundaries. In M. Wozny, J. Turner, and K. Preiss, editors, *Geometric Modeling for Product Engineering*, pages 145–180. IFIP, Elsevier Science Publishers B.V. (North-Holland), 1990.

[16] J. Rossignac and A. Requicha. Constructive non-regularized geometry. *Computer Aided Design*, 23(2):21–32, 1991.

[17] V. Shapiro. Real functions for representation of rigid solids. *Computer Aided Geometric Design*, 11(2):153–175, 1994.

[18] V. Shapiro. Maintenance of geometric representations through space decompositions. *International Journal of Computational Geometry and Applications*, 7(4):383–418, 1997.

[19] L. van den Dries. A generalization of the Tarski-Seidenberg theorem, and some nondefinability results. *Bulletin of the AMS*, 15(2):189–193, 1986.

[20] L. van den Dries. *Tame Topology and O-minimal Structures*. LMS Lecture Note Series, Vol 248. Cambridge University Press, 1998.

[21] L. vandenDries and C. Miller. Geometric categories and o-minimal structures. *Duke Math*, 84:497–540, 1996.

[22] C. Wall. Regular stratifications. In A. Dold and B. Eckman, editors, *Dynamical Systems*, volume 468 of *Lecture Notes in Mathematics*, pages 332–344. Springer-Verlag, Berlin, 1975. (Proceedings of a Symposium held at the University of Warwick 1973/1974).

[23] H. Whitney. Elementary structure of real algebraic varieties. *Annals of Mathematics*, 66(3):545–556, November 1957.

[24] A. Wilkie. Model completeness results for expansions of the real field by pfaffian functions and the exponential function. *Journal of the AMS*, 9:1051–1094, 1996.