# THE NON-COMPACTNESS OF SQUARE.

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## 1. Introduction

This note gives proves two theorems. The first is that it is consistent to have $\square_{\omega_n}$ for every $n$, but not have $\square_{\aleph_\omega}$. This is done by carefully collapsing a supercompact cardinal and adding square sequences to each $\omega_n$. The crux of the proof is that in the resulting model every stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega)$ *reflects* to an ordinal of cofinality $\omega_1$, that is to say it has stationary intersection with such an ordinal.

This result contrasts with compactness properties of square shown in [3]. In that paper it is shown that if one has square at every $\omega_n$, then there is a square type sequence on the points of cofinality $\omega_k$, $k > 1$ in $\aleph_{\omega+1}$. In particular at points of cofinality greater than $\omega_1$ there is a strongly non-reflecting stationary set of points of countable cofinality.

The second result answers a question of Džamonja, by showing that there can be no squarelike sequence above a supercompact cardinal, where "squarelike" means that one replaces the requirement that the cofinal sets be closed and unbounded by the requirement that they be stationary at all points of uncountable cofinality.

## 2. Some Lemmas

In this section we define a forcing notion and show some lemmas. The forcing notion is a standard style of Namba forcing and the lemmas are standard. We prove them here for the benefit of the reader.

Let $n \to (n_0, n_1)$ be a bijection from $\omega$ to $\omega \times (\omega \setminus \{0,1\})$. We say that a tree is *standard* for our partial ordering iff

1. $T \subset (\aleph_\omega)^{<\omega}$
2. For all $\sigma \in T$ and $n \in \mathrm{dom}(\sigma), \sigma(n) \in \omega_{n_1}$
3. There is a maximal $\sigma \in T$ (called the *stem* of $T$) such that for all $\tau \in T, \sigma \subset \tau$ or $\tau \subset \sigma$.
4. For all $\tau$ extending the stem $\sigma$, if $n \in \mathrm{dom}(\tau), n \supset \mathrm{dom}(\sigma)$ then $|\{\alpha : \tau \restriction n^\frown\alpha \in T\}| = \omega_{n_1}$

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Our partial ordering $\mathcal{P}$ will consist of standard trees with the ordering of inclusion.

It is easy to verify the following facts:

- $|\mathcal{P}| = |\prod_{n\in\omega} 2^{\omega_n}|$.
- If $G \subset \mathcal{P}$ is generic then $\bigcup\bigcap G$ is a function $f : \omega \to \aleph_\omega$ such that for all $n, f(n) \in \omega_{n_1}$ and for all $m, \{f(n) : n_1 = m\}$ is cofinal in $\omega_m$.

We will say that a sequence of standard trees $\langle T_n : n \in \omega\rangle$ is a *fusion sequence* if $T_{n+1} \subseteq T_n$ for all $n$, all the trees have the same stem $\sigma$, and for every $n$ and every $\tau \in T_n$ with $\mathrm{dom}(\tau) \leq \mathrm{dom}(\sigma) + n + 1$ we have $\tau \in T_{n+1}$. It is easy to see that if $\langle T_n : n \in \omega\rangle$ is a fusion sequence then there is a standard tree $T_\infty$ with stem $\sigma$ such that $T_\infty \leq T_n$ for all $n < \omega$.

We will need the following lemmas. The first is a standard forcing exercise:

**Lemma 1.** *Let $\dot\tau$ be a $\mathcal{P}$-term for an ordinal less than $\aleph_{\omega+1}$ and $T$ be a standard tree. Then there is a standard tree $T' \subset T$ with the same stem as $T$ and an $n$ such that for all $\sigma \in T'$ of length $n$, if we let $T'_\sigma = \{\sigma' : \sigma' \supset \sigma$ or $\sigma' \subset \sigma\}$, then for some $\beta < \aleph_{\omega+1}, T'_\sigma \Vdash \dot\tau = \beta$.*

From this we can see:

**Lemma 2.** *Suppose that $G \subset \mathcal{P}$ is generic. Then:*

1. $V[G] \models cf(\aleph^V_{\omega+1}) \geq \omega_1$.
2. *If $S \subset \aleph_{\omega+1} \cap cof(\omega)$ is a stationary set in $V$, then $S$ is stationary in $V[G]$.*

$\vdash$ Let $\langle\dot\alpha_n : n \in \omega\rangle$ be a term for an $\omega$-sequence of ordinals and let $T$ be an arbitrary standard tree. Repeatedly applying Lemma 1, we can build a standard tree $T' \subset T$ with the same stem as $T$ with the property that for all infinite branches $b$ through $T'$, there is a sequence of natural numbers $\langle m_n : n \in \omega\rangle$ and ordinals $\langle\beta_n : n \in \omega\rangle$ such that if $T'_{b\restriction m_n} = \{\sigma \in T' : b \restriction m_n \subset \sigma$ or $\sigma \subset b \restriction m_n\}$ then

$$T'_{b\restriction m_n} \Vdash \dot\alpha_n = \check\beta_n$$

This property is absolute between $V$ and the generic extension determined by $\mathcal{P}$, so if $f = \bigcup\bigcap G$ we know that there is such a sequence for $f$ in $V[G]$.

Fix now any collection of terms $\{\dot\alpha_n : n \in \omega\}$. Define $T'$ as in the last paragraph and let $A = \{\beta :$ there is a $\sigma \in T'$ and some $n, T'_\sigma \Vdash \dot\alpha_n = \check\beta\}$. Then $|A| = \aleph_\omega$ and $T' \Vdash \{\dot\alpha_n : n \in \omega\} \subset A$. Thus we have established claim 1 of the lemma.

In particular, we have shown that if $\dot{\alpha}$ is a term for an ordinal less than $\aleph_{\omega+1}$ and $T$ is a condition then there is a $T' \subset T$ with the same stem as $T$ and a $\gamma < \aleph_{\omega+1}$ with the property that $T' \Vdash \dot{\alpha} < \gamma$.

Let $\dot{C}$ be an arbitrary $\mathcal{P}$-term for a closed unbounded subset of $\aleph_{\omega+1}$, and $S$ be a stationary subset of $\aleph_{\omega+1} \cap \mathrm{cof}(\omega)$. Let $N \prec \langle H(\lambda), \in, <_\lambda, \dot{C}, \mathcal{P}\rangle$ for some large $\lambda$ be such that $|N| = \aleph_\omega$ and $\gamma^* = N \cap \aleph_{\omega+1} \in S$, where $<_\lambda$ is a well-ordering of $H_\lambda$. Choose a sequence $\langle \gamma_n : n \in \omega\rangle$ converging to $\gamma^*$. Let $\dot{\alpha}_n \in N$ be a $\mathcal{P}$-term for an element of $\dot{C}$ above $\gamma_n$.

Using this one can inductively build a fusion sequence of trees $\langle T_n : n \in \omega\rangle$ such that:

- $T_n \in N$
- $T_n \Vdash \dot{\alpha_n} = \delta_n$.

There is a $T^* \in \mathcal{P}$ such that for all $n, T^* \leq T_n$. Clearly $T^* \Vdash \sup\{\dot{\alpha}_n\} = \gamma^*$ and hence $T^* \Vdash \dot{C} \cap S \neq \emptyset$. ⊣

We do not know whether forcing with $\mathcal{P}$ can ever preserve $\aleph^V_{\omega+1}$. Results of Shelah imply that if $\square^*_{\aleph_\omega}$ holds in $V$ then $\mathcal{P}$ will collapse $\aleph^V_{\omega+1}$. More information about this problem may be found in [1].

In the same spirit as the previous lemma, but with a different proof we have:

**Lemma 3.** *Forcing with $\mathcal{P}$ preserves stationary subsets of $\aleph_1$.*

There are many places to see the proof of this, for example we cite Namba's original articles [9, 8], Shelah's book [10] or [5].

We will also need some standard partial orderings for forcing square. There are several variations on these; basic properties are discussed in [2] among other places. In this paper we will force with partial orderings that add square by initial segments; this partial ordering for adding a square sequence was introduced by Jensen.

**Definition 4.** *Let $\kappa$ be an uncountable cardinal. Let $\mathbb{P}(\kappa)$ be the partial ordering consisting of all sequences $\langle C_\alpha : \alpha \leq \delta\rangle$ for some $\delta \in \kappa^+$, that have the following properties:*

1. *The set $C_\alpha$ is closed and unbounded in $\alpha$.*
2. *The order type of $C_\alpha$ is less than or equal $\kappa$.*
3. *If $\beta$ is a limit point of $C_\alpha$, then $C_\beta = C_\alpha \cap \beta$.*

*The ordering of $\mathbb{P}(\kappa)$ is extension.*

Proofs of the following facts can be found in [2]:

- The poset $\mathbb{P}(\kappa)$ is countably closed. (In particular forcing with $\mathbb{P}(\kappa)$ preserves stationary subsets of $\operatorname{cof}(\omega) \cap \lambda$ for any regular cardinal $\lambda$.)
- The poset $\mathbb{P}(\kappa)$ has the property that player II wins the game $G^{\mathrm{II}}_{\kappa^+}$ as defined in [4].
- For all $\delta \in \kappa^+$ the collection of conditions $p \in \mathbb{P}(\kappa)$ which are of length at least $\delta$ is a dense set.
- Forcing with $\mathbb{P}(\kappa)$ adds a $\square_\kappa$ sequence, without collapsing cardinals, or adding any $\kappa$-sequences.

In our situation we will be adding square successively to each $\omega_n, n \geq 1$. With this in mind we define a sequence of partial orderings $Sq(n)$ for $n \in \omega + 1 \backslash \{0\}$ by induction on $n$.

(1) Let $Sq(1) = \mathbb{P}(\omega_1)$.
(2) $Sq(n+1) = Sq(n) * \mathbb{P}(\omega_{n+1})$.
 Finally we define
(3) $Sq(\omega)$ to be the inverse limit of the sequence $\langle Sq(n) : n \in \omega \rangle$.

It is then standard to show that the iteration $Sq(\omega)$ is $\omega$-closed, and for each $n, Sq(\omega) \sim Sq(n) * \mathbb{R}$, where $\mathbb{R}$ is $\omega_{n+1}$-strategically closed. In particular, forcing with $Sq(\omega)$ preserves cardinals and cofinalities, and all stationary subets of $\lambda \cap \operatorname{cof}(\omega)$ for any uncountable regular $\lambda$. Moreover $Sq(\omega)$ adds $\square_{\omega_n}$ sequences for $n \geq 1$ and a generic object $H \subset Sq(\omega)$ corresponds exactly to a sequence $\langle H_n : n \in \omega \rangle$, with the property that $H_{n+1} \subset \mathbb{P}(\omega_{n+1})$ is generic over $V[\langle H_k : 1 \leq k \leq n \rangle]$ for each $n$.

## 3. The Forcing

In this section we prove:

**Theorem 5.** *Suppose that $\kappa$ is a supercompact cardinal. Then there is a forcing extension in which:*

(1) $\kappa = \aleph_2$
(2) *For all $n \in \omega \backslash \{0\}, \square_{\omega_n}$ holds.*
(3) *Every stationary subset of $\aleph_{\omega+1} \cap$ cof$(\omega)$ reflects to an $\alpha \in$ cof$(\omega_1)$.*

$\vdash$ We follow the outline of the proof of the consistency of Martin's Maximum. (Or simply use the Martin's Maximum partial ordering directly, with slightly more argument.) By the main result of [6], we fix a function $f : \kappa \to \kappa$ such that for every ordinal $\gamma$ there is a $\gamma^+$-supercompact embedding $j : V \to M$ with critical point $\kappa$, such that $j(f)(\kappa) = \gamma$.

A typical component of our forcing is the partial ordering $Sq(\omega) * \mathcal{P}$ (as defined in the previous section.) Since $Sq(\omega)$ is $\omega$-closed and $\mathcal{P}$ preserves stationary subsets of $\omega_1$ we see that this two-step iteration preserves stationary subsets of $\omega_1$.

We define a semi-proper iteration $\mathbb{P} = \langle(\mathbb{P}_\alpha, \mathbb{Q}_\alpha) : \alpha < \kappa\rangle$ of length $\kappa$ with revised countable support so that:

(1) $\mathbb{Q}_\alpha = \mathbf{1}$ unless $\alpha$ is inaccessible and for all generic $G_\alpha \subset \mathbb{R}_\alpha$, $\alpha$ is the $\omega_2$ of $V[G_\alpha]$.
   In this case
(2) If $f(\alpha) = 0$ then if
$$V[G_\alpha] \models Sq(\omega) * \mathcal{P} \text{ is semiproper}$$
   we let $\mathbb{Q}_\alpha = Sq(\omega) * \mathcal{P} * Col(\omega_1, \aleph_{\omega+1}^{V[G_\alpha]})$, otherwise $\mathbb{Q}_\alpha = \mathbf{1}$.
(3) If $f(\alpha) > 0$ then we let $\mathbb{Q}_\alpha = Col(\omega_1, f(\alpha))$.

Exactly as in the proof of the consistency of Martin's Maximum that appeared in [7], we see that the revised countable support limit at stage $\kappa, \mathbb{R}_\kappa$, is $\kappa$ chain condition, makes $\kappa$ into $\omega_2$ and is semi-proper. (In particular, in $V[G_\kappa], \kappa^{+\omega+1} = \aleph_{\omega+1}$.) Moreover, if $G_\kappa$ is generic over $\mathbb{R}_\kappa$ then by Lemma 3 from [7]:
$$V[G_\kappa] \models \text{ for all partial orderings } \mathbb{P}, \mathbb{P} \text{ is semi-proper iff } \mathbb{P} \text{ preserves stationary subsets of } \omega_1$$

Let $G_\kappa \subset \mathbb{R}_\kappa$ be generic and let $H \subset Sq(\omega)^{V[G_\kappa]}$ be generic. We claim that $V[G_\kappa, H]$ satisfies the conclusion of the theorem. Since $Sq(\omega)$ adds a square sequence to each $\aleph_n$ and preserves $\aleph_{\omega+1}$ the only part of the conclusion left to verify is that every stationary subset of $\aleph_{\omega+1}$ reflects.

**Claim 6.** *Fix a $\kappa^{+\omega+1}$-supercompact embedding $j : V \to M$ such that $j(f)(\kappa) = 0$. Then:*

(1) *There are generic $G', H'$ such that the embedding $j$ can be extended to a $\hat{\jmath} : V[G_\kappa, H] \to M[G', H']$.*
(2) *If $S \subset \kappa^{+\omega+1} \cap cof(\omega)$ is stationary in $V[G_\kappa, H]$, then $S$ is stationary in $V[G', H']$.*

$\vdash$ Standard large cardinal technology shows that if we take any $V$-generic $G' \subset j(\mathbb{R})$ extending $G_\kappa$, then $j$ extends to a generic embedding:
$$j_0 : V[G_\kappa] \to M[G'].$$
Since in $V[G_\kappa]$ semi-properness is equivalent to preserving stationary subsets of $\omega_1$, the partial ordering $Sq(\omega) * \mathcal{P}$ is semi-proper. Since $j$ is $\kappa^{\omega+1}$-supercompact, this is absolute between $V[G_\kappa]$ and $M[G_\kappa]$. Hence, in $M[G_\kappa]$, the partial ordering $\mathbb{Q}_\kappa$ is taken to be $Sq(\omega) * \mathcal{P} *$

$Col(\omega_1, \kappa^{+\omega+1})$. In particular $G' = G_\kappa * H * P * C * G^*$, where $H * P * C \subset Sq(\omega) * \mathcal{P} * Col(\omega_1, \kappa^{+\omega+1})$ is generic over $V[G_\kappa]$.

We must build a master condition $m \in j(Sq(\omega))^{M[G_\kappa]}$ with the property that for all $q \in H, m \leq j(q)$. Then for any $M[G']$-generic $H'$ with $m \in H'$ there is an elementary

$$\hat{\jmath} : V[G_\kappa, H] \to M[G', H'].$$

For each $n \in \omega \backslash \{0\}$ let $\gamma_n = \sup(j``\kappa^{+n})$. In $M[G']$ each $\gamma_n$ has cofinality $\omega$. Choose a cofinal $\omega$-sequence $C_n$ in each $\gamma_n$. If $H \sim \langle H_n : n \in \omega \rangle$ where $H_n$ is the generic square sequence through $\kappa^{+n}$, then we let $m_1$ be the canonical term for $\bigcup j``H_1 \cup \{(\gamma_1, C_1)\}$. It is then easy to verify by induction on $n$ that:

- If $(m_1, m_2, \dots m_n) \in H_1' * H_2' * \dots H_n'$ and $H_1' * H_2' * \dots H_n'$ are generic for $Sq(n)$ then $j$ extends to an elementary embedding

  $$j_n : V[G_\kappa * H_1 * H_2 * \dots H_n] \to M[G' * H_1' * H_2' * \dots H_n'].$$

- $m_{n+1} =_{def} j_n``H_{n+1} \cup \{(\gamma_{n+1}, C_{n+1})\}$ is a condition in the partial ordering for adding $\square_{j(\kappa^{+n})}$ that is stronger than each element of $j``H_{n+1}$.

(In the jargon, "we take the union of the image of the generic object at $\kappa_n$ and put the sequence $C_n$ on top".)

If we let $m$ be the sequence of $M[G']$ terms for $\langle m_n : n \in \omega \rangle$, then we see that $m$ is the desired master condition.

We now show the second part of Claim 6. Let $S \subset \kappa^{+\omega+1} \cap \mathrm{cof}(\omega)$ be stationary. Then by Lemma 2, $S$ is stationary in $V[G_\kappa * H * P]$. Since $S \subset \mathrm{cof}(\omega)$, $S$ remains stationary after one forces with $Col(\omega_1, \kappa^{+\omega+1})$ and hence is stationary in $V[G * H * P * C]$. Since $\kappa^{+\omega+1}$ is of cofinality $\omega_1$ in this model, if $D \subset \kappa^{+\omega+1}$ is any closed unbounded set of order type $\omega_1$, preserving the stationarity of $S$ is equivalent to preserving the stationary subset $T$ of $\omega_1$ determined by $D \cap S$. The forcing that produces $G^*$ over $V[G * H * P * C]$ is semi-proper and hence preserves the stationarity of $T$. Finally $Sq(\omega)$ is $\omega$-closed over $V[G'] = V[G_\kappa * H * P * C * G^*]$ and hence preserves the stationarity of $T$. This proves the claim. $\dashv$

**Claim 7.** *Let $\gamma = \sup\ j``\kappa^{+\omega+1}$ and let $S \subset \kappa^{+\omega+1} \cap cof(\omega)$ be stationary in $V[G, H]$. Then $j``S$ is stationary in $\gamma$ in the model $V[G', H']$.*

$\vdash$ Since $\gamma$ has cofinality $\omega_1$ in the model and the forcing for producing $G'$ over the model $V[G_\kappa * H * P]$ is semi-proper it suffices to show that $j``S$ is stationary in $\gamma$ in the model $V[G_\kappa * H * P]$. We work in the model $V[G_\kappa * H]$.

Let $\dot{C}$ be a term for a closed unbounded subset of $\gamma$ lying in $V[G_\kappa * H]^{\mathcal{P}}$. Consider $\dot{C}' = \{\alpha \in \kappa^{+\omega+1} : j(\alpha) \in C\}$. We now argue that $V[G_\kappa * H]^{\mathcal{P}} \models C' \cap \mathrm{cof}(\omega)^{V[G_\kappa,H]}$ is closed unbounded relative to $(\kappa^{+\omega+1})^V \cap \mathrm{cof}(\omega)^{V[G_\kappa,H]}$. This suffices, since forcing with $\mathcal{P}$ preserves stationary subsets of $\kappa^{+\omega+1} \cap \mathrm{cof}(\omega)$. In particular, there is a $\delta \in C' \cap S$. Then $j(\delta) \in j(S) \cap j``\lambda$.

The argument that $V[G_\kappa * H]^{\mathcal{P}} \models C' \cap \mathrm{cof}(\omega)^{V[G_\kappa,H]}$ is closed unbounded relative to $(\kappa^{+\omega+1})^V \cap \mathrm{cof}(\omega)^{V[G_\kappa,H]}$ follows the argument for Lemma 2. That $C' \cap \mathrm{cof}(\omega)^{V[G_\kappa,H]}$ is relatively closed is clear, since $j$ is continuous at points of cofinality $\omega$ in $V[G_\kappa * H]$.

We show that $C' \cap \mathrm{cof}(\omega)^{V[G_\kappa,H]}$ is unbounded. For each ordinal $\alpha \in \kappa^{+\omega+1}$, let $\dot{\beta}(\alpha)$ be the $\mathcal{P}$ term for the least ordinal $\beta$ in $\kappa^{+\omega+1}$ such that there is a $\delta \in \dot{C}$ with $j(\alpha) \leq \delta \leq \beta$.

Let $N \prec \langle H(\lambda), \in, <_\lambda, \dot{C}, \mathcal{P}\rangle$ for some large $\lambda$ be such that $|N| = \aleph_\omega$ and $\gamma^* = N \cap \aleph_{\omega+1} \in \mathrm{cof}(\omega)$. Choose a sequence $\langle \gamma_n : n \in \omega\rangle$ converging to $\gamma^*$. As in Lemma 2, we can build a fusion sequence of trees $\langle T_n : n \in \omega\rangle$ such that:

- $T_n \in N$
- $T_n \Vdash \dot{\beta}(\gamma_n) = \delta_n$.
- There is a $T^* \in \mathcal{P}$ such that for all $n, T^* \leq T_n$.

Then $T^* \Vdash \sup\{\dot{\beta}(\gamma_n)\} = \gamma^*$. Again, since $j$ is continuous at points of cofinality $\omega$, $j(\gamma)^* = \sup j``\gamma^*$. In particular, $T^* \Vdash j(\gamma^*) \in \dot{C}$.

Since $N$ was arbitrary, this shows the claim ⊣

The claim suffices to prove the theorem, since $j``S \subset j(S)$, and thus $M[G', H'] \models j(S)$ reflects to $\gamma$. By elementarity, we see that $V[G_\kappa, H] \models \exists\gamma(S$ reflects to $\gamma)$.

This finishes the claim and the proof of Theorem 5. ⊣

## 4. A remark about squares above a supercompact cardinal

In this section we answer a question of Džamonja by showing the following:

**Theorem 8.** *Let $\kappa$ be a supercompact cardinal and $\lambda \geq \kappa$ a regular cardinal. Then there is no sequence of sets $\langle S_\alpha : \alpha < \lambda^+\rangle$ such that:*

1. *$S_\alpha \subset \alpha$ is unbounded in $\alpha$*
2. *o.t. $S_\alpha \leq \lambda$*
3. *If $cf(\alpha) > \omega$ then $S_\alpha$ is stationary in $\alpha$*
4. *If $\beta \in S_\alpha$ is a limit point of $S_\alpha$, then $S_\beta = S_\alpha \cap \beta$.*

We note that this result slightly generalizes a result of Solovay that there can be no square sequence above a supercompact cardinal.

$\vdash$ (Theorem 8) Suppose that there is such a sequence, $\langle S_\alpha : \alpha < \lambda^+ \rangle$. We follow the outline of Solovay's argument with a slight additional twist.

Let $j : V \longrightarrow M$ be an elementary embedding witnessing that $\kappa$ is $\lambda^+$-supercompact; that is to say $j$ has critical point $\kappa$, $j(\kappa) > \lambda^+$ and ${}^{\lambda^+}M \subseteq M$. We set $\gamma = \sup j``\lambda^+$. Since $j``\lambda^+ \in M$ it is easy to see that $\gamma$ has cofinality $\lambda^+$ both in $V$ and in the inner model $M$; since $\lambda^+ < j(\kappa) < j(\lambda^+)$ and $j(\lambda^+)$ is regular in $M$, it follows that $\gamma < j(\lambda^+)$.

Let $\langle T_\beta : \beta < j(\lambda^+) \rangle = j(\langle S_\alpha : \alpha < \lambda^+ \rangle)$, and let $U = T_\gamma$. By elementarity, in $M$ the set $U$ is a stationary subset of $\gamma$.

Let $U^*$ be the set of those $\mu < \gamma$ such that $\mu \in U$, and both $U \cap \mu$ and $j``\lambda^+ \cap \mu$ are unbounded in $\mu$. It is clear that in $M$ the set $U^*$ is stationary in $\gamma$. We note that if $\mu \in U^*$ then $\mu$ is a point in $T_\gamma$ which is a limit point of $T_\gamma$, so by elementarity and the coherence property (4) we have $U \cap \mu = T_\mu$.

**Claim 9.** *$U^* \cap j``\lambda^+$ is unbounded in $\gamma$.*

$\vdash$ We work in $M$. Given $\alpha < \gamma$, let $\beta \in U^*$ be least with $\beta > \alpha$. We claim that $\beta$ has cofinality $\omega$. Suppose for a contradiction that $\beta$ has uncountable cofinality. Then since $U \cap \beta = T_\beta$, it follows by elementarity that $U \cap \beta$ is stationary in $\beta$. Since $U$ and $j``\lambda^+$ are both unbounded in $\beta$, we may find $\beta^* \in U^*$ with $\alpha < \beta^* < \beta$. This contradicts the minimal choice of $\beta$.

The elementary embedding $j$ is continuous at points of cofinality less than $\kappa$, and since $j``\lambda^+$ is unbounded in $\beta$ it follows that $\beta \in j``\lambda^+$. So $U^* \cap j``\lambda^+$ is unbounded in $\gamma$, as claimed. $\dashv$

Let $X$ be the unbounded subset of $\lambda^+$ consisting of those $\eta$ such that $j(\eta) \in U^*$. Find $\eta \in X$ such that the order type of $X \cap \eta$ is greater than $\lambda$. Since $j(\eta) \in U^*$, we see that $U \cap j(\eta) = T_{j(\eta)} = j(S_\eta)$.

For every $\zeta \in X \cap \eta$, $j(\zeta) \in U \cap j(\eta)$ and so by elementarity $\zeta \in S_\eta$. So $X \cap \eta \subseteq S_\eta$ and thus the order type of $S_\eta$ is greater than $\lambda$, contradicting our assumptions about the sequence $\langle S_\alpha : \alpha < \lambda^+ \rangle$. $\dashv$

**Remark:** Martin Zeman has pointed out that if $\kappa$ is subcompact then this suffices to prove Theorem 8 in the case when $\lambda = \kappa$. The cardinal $\kappa$ is *subcompact* if for every $A \subseteq H_{\kappa^+}$ there exist $\delta < \kappa$, $a \subseteq H_{\delta^+}$ and an elementary $\pi : (H_{\delta^+}, \in, a) \to (H_{\kappa^+}, \in, A)$ such that the critical point of $\pi$ is $\delta$ and $\pi(\delta) = \kappa$.

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