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George-Veeramani Fuzzy Metrics Revised

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Abstract: In this note, we present an alternative approach to the concept of a fuzzy metric, calling it a revised fuzzy metric. In contrast to the traditional approach to the theory of fuzzy metric spaces which is based on the use of a t-norm, we proceed from a t-conorm in the definition of a revised fuzzy metric. Here, we restrict our study to the case of fuzzy metrics as they are defined by George-Veeramani, however, similar revision can be done also for some other approaches to the concept of a fuzzy metric.

Keywords: fuzzy metric; t-norm; t-conorm

1. Introduction and Motivation

In 1951, Menger introduced the concept of a statistical metric [1]. Based on the concept of a statistical metric, Kramosil and Michalek introduced the notion of a fuzzy metric in [2]. Here, we call it a KM-*fuzzy metric*. A KM-fuzzy metric is, in a certain sense, equivalent to a statistical metric, but there are essential differences in their definitions and interpretations. In 1994, George and Veeramani [3], see also [4], slightly modified the original concept of a KM-fuzzy metric—we call this modification by a GV-*fuzzy metric*. This modification allows many natural examples of fuzzy metrics, in particular, fuzzy metrics constructed from metrics. GV-fuzzy metrics appear to be more appropriate also for the study of induced topological structures.

Along with the principal interest of many researchers in the theoretical aspects of the theory of fuzzy metrics—in particular, the topological and sequential properties of fuzzy metric spaces, their completeness, fixed points of mappings, etc.—fuzzy metrics have also aroused interest among specialists working in various applied areas of mathematics. Among others, fuzzy metrics have been used in decision making problems with uncertain and imprecise information and other engineering problems. For example, Niskanen [5] developed a fuzzy metric-based reasoning approach to decision making in the problems of soft computing Gregori et al. [6] constructed a fuzzy metric that simultaneously takes into account two different distance criteria between color image pixels and used this fuzzy metric to filter noisy images, etc.

The fact that KM- and GV-fuzzy metrics are obtained on the basis of a statistical metric is reflected in the assumption that the degree of closeness of two points in a fuzzy metric space corresponds to the probability of the "coincidence" of these points in the statistical metric. In particular, the fuzzy distance between two equal points is 1, while in cases when one point is "far" from the other, the fuzzy distance between them is "close" to 0. This may look strange if only one does not think of a fuzzy metric as the counterpart of a statistical metric. In these notes, we consider how the definition of a fuzzy metric can be revised in order to get a notion which is better coordinated with the intuitive meaning of a distance.

Here, we restrict ourselves to the case of GV-fuzzy metrics. Obviously, a similar revision can be done for KM-fuzzy metrics as well.

2. GV-Fuzzy Metrics Revised

First, we recall the notion of a fuzzy metric as it was defined by George and Veeramani in [3,4]. Let X be a set, $* : [0,1] \times [0,1] \to [0,1]$ be a continuous t-norm, and $\mathbb{R}^+ = (0, +\infty)$. A GV-fuzzy metric on X is a pair ($(m, *)$, or simply m), where the mapping $m : X \times X \times \mathbb{R}^+ \to (0,1]$ satisfies the following conditions for all $x, y, z \in X$, $s, t \in \mathbb{R}^+$:

(1GV) $m(x,y,t) > 0$;
(2GV) $m(x,y,t) = 1 \iff x = y$;
(3GV) $m(x,y,t) = m(y,x,t)$;
(4GV) $m(x,z,t+s) \geq m(x,y,t) * m(y,z,s)$;
(5GV) $m(x,y,-) : \mathbb{R}^+ \to [0,1]$ is continuous.

If $(m, *)$ is a GV-fuzzy metric on X, then the triple $(X, m, *)$ is called a GV-fuzzy metric space.

Let $^c : [0,1] \to [0,1]$ be an order reversing involution on $[0,1]$. In particular, ^{c} can be the standard involution, that is, $a^c = 1 - a$ for all $a \in [0,1]$. Now, given a GV-fuzzy metric $m : X \times X \times (0,+\infty) \to (0,1]$, let the mapping $m^c : X \times X \times (0,+\infty) \to [0,1)$ be defined by $m^c(x,y,t) = (m(x,y,t))^c$. Then, the properties (1GV)–(5GV) correspond to the following properties of the mapping (m^c) holding for all $x, y \in X$ and for all $t, s > 0$:

(1GVc) $m^c(x,y,t) < 1$;
(2GVc) $m^c(x,y,t) = 0 \iff x = y$;
(3GVc) $m^c(x,y,t) = m^c(y,x,t)$;
(4GVc) $m^c(x,z,t+s) \leq m^c(x,y,t) \oplus m^c(y,z,s)$;
(5GVc) $m^c(x,y,-) : \mathbb{R}^+ \to [0,1]$ is continuous,

where $\oplus$ is the t-conorm corresponding to the t-norm $*$ and involution ^{c}, that is $a \oplus b = (a^c * b^c)^c$ for all $a, b \in [0,1]$. Conversely, if a mapping $\mu : X \times X \times (0,+\infty) \to [0,1]$ satisfies properties (1GVc)–(5GVc), then by setting $\mu^c = m$, we get a GV-fuzzy metric ($\mu^c : X \times X \times \mathbb{R}^+ \to [0,1]$) for the t-norm corresponding to the t-conorm $\oplus$ and involution $^c : [0,1] \to [0,1]$. This observation leads us to the following definition.

Definition 1. *Let X be a set and $\oplus : [0,1] \times [0,1] \to [0,1]$ be a continuous t-conorm. A revised GV-fuzzy metric or an RGV-fuzzy metric for short, on the set X is a pair $(\mu, \oplus)$ or simply μ, where the mapping $\mu : X \times X \times \mathbb{R}^+ \to [0,1)$ satisfies the following conditions for all $x, y, z \in X$, $s, t \in \mathbb{R}^+$:*

(1RGV) $\mu(x,y,t) < 1$;
(2RGV) $\mu(x,y,t) = 0 \iff x = y$;
(3RGV) $\mu(x,y,t) = \mu(y,x,t)$;
(4RGV) $\mu(x,z,t+s) \leq \mu(x,y,t) \oplus \mu(y,z,s)$;
(5RGV) $\mu(x,y,-) : \mathbb{R}^+ \to [0,1)$ *is continuous.*

From the above consideration we get the following theorem.

Theorem 1. *GV-fuzzy metrics and RGV-fuzzy metrics are equivalent concepts.*

Remark 1. It is well-known that GV-fuzzy metrics are non-decreasing in the third variable. From here, or independently, by analyzing the definition of an RGV-fuzzy metric, we conclude that RGV-*fuzzy metrics are non-increasing in the third variable.* This allows us to give the following visual interpretation of an RGV-fuzzy metric. Assume that we are looking from a distance ($t \in (0,+\infty)$) at a plane filled up with pixels. We estimate the distance between pixels x and y by means of an RGV-fuzzy metric $\mu(x,y,t)$. Being close to the plane, we see quite clearly how far the two pixels x and y are. However, going further from the plane, our ability to distinguish the real distance between different pixels becomes weaker and, at some moment, two different pixels can merge into one in our eye-pupil.

Remark 2. The definition of an RGV-fuzzy metric becomes especially visual in case of a Łukasiewicz t-conorm ($a \oplus b = \min\{a+b, 1\}$) since in this case the third axiom in its definition

$$\mu(x,z,t+s) \leq \mu(x,y,t) + \mu(y,z,s)$$

is an obvious generalization of the triangular axiom in the definition of a metric:

$$d(x,z) \leq d(x,y) + d(y,z).$$

Remark 3. A natural generalization of the definition of an RGV-fuzzy metric can be obtained by allowing the mapping μ to take any non-negative numbers and not being restricted to [0,1). In this case, an RGV-fuzzy metric is defined as a mapping ($\mu : X \times X \times \mathbb{R}^+ \to [0,\infty)$) satisfying properties (2RGV), (3RGV), (4RGV), and (5RGV) of Definition 1. However, we will not go further into this here, since the main aim of these notes is just to present an alternative view on GV-fuzzy metrics. Note only that a similar extension would not be possible in the context of ordinary GV-fuzzy metrics.

3. Some Remarks on the Theory of RGV-Fuzzy Metrics

Since RGV-fuzzy metrics are equivalent to GV-fuzzy metrics, the theories based on these concepts are equivalent. The difference is in the definitions, the proofs, and the interpretations of results. In this section, we give some illustrations of this situation. Namely, we revise a part of the original paper by George and Veeramani [3] in the context of RGV-fuzzy metrics.

Example 1. In the theory of GV-fuzzy metrics, the standard fuzzy metric induced by an ordinary metric ($d : X \times X \to [0,+\infty)$) plays an important role. In the context of RGV-fuzzy metrics, we have the corresponding standard RGV-fuzzy metric induced by a metric (d). It is defined by

$$\mu(x,y,t) = \tfrac{d(x,y)}{t+d(x,y)}.$$

Note that it is an RGV-fuzzy metric for the maximum t-conorm $\oplus$ (that is, $a \oplus b = a \vee b$) and hence, is applicable for any other t-conorm.

Definition 2. *Given an RGV-fuzzy metric space ($(X,\mu,\oplus)$), we define the open ball ($B(x,r,t)$) with center $x \in X$, radius $r \in (0,1)$, and at the level of $t > 0$ as*

$$B(x,r,t) = \{y \in X : \mu(x,y,t) < r\}.$$

Proposition 1. *An open ball ($B(x,r,t) = \{y \in X : \mu(x,y,t) < r\}$) is "indeed open" in the sense that for any $y \in B(x,r,t)$, $t_0 > 0$ and $\varepsilon \in (0,1)$ exist such that $B(y,\varepsilon,t_0) \subseteq B(x,r,t)$.*

Proof. Since $y \in B(x,r,t)$, it follows that $\mu(x,y,t) < r$. By continuity of μ in the third argument, we can find $t_1 < t$, such that $\mu(x,y,t_1) < r$. Let $r_0 = \mu(x,y,t_1)$. Since $r_0 < r$, by continuity of the t-conorm $\oplus$, we can find $\varepsilon \in (0,1)$ such that $r_0 \oplus \varepsilon < r$. Let $t_0 = t - t_1$. We claim that $B(y,\varepsilon,t_0) \subseteq B(x,r,t)$.

Indeed, if $z \in B(y,\varepsilon,t_0)$, then $\mu(y,z,t_0) < \varepsilon$ and hence, since $t = t_1 + t_0$, we have by axiom (4RGV):

$$\mu(x,z,t) \leq \mu(x,y,t_1) \oplus \mu(y,z,t_0) \leq r_0 \oplus \varepsilon < r,$$

that is, $B(y,\varepsilon,t_0) \subseteq B(x,r,t)$. □

From this proposition, we get an important corollary.

Corollary 1. *Given an RGV-fuzzy metric space ((X,μ)), the family*

$$\mathcal{B} = \{B(x,r,t) : x \in X, r \in (0,1), t \in (0,+\infty)\}$$

is a base of some topology (τ) on the set (X). We call it the topology induced by an RGV-fuzzy metric μ.

Noticing that the family $\{B(x, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base in the point $x \in X$, we get the following.

Proposition 2. *The topology induced by a RGV-fuzzy metric is first countable.*

Proposition 3. *The topology induced by an RGV-fuzzy metric is Hausdorff.*

Proof. Let (X, μ) be an RGV-fuzzy metric space, let $x, y \in X, x \neq y$, and let $\mu(x, y, t) = r$. Since $r > 0$ and t-conorm is continuous, we can find $\varepsilon > 0$ such that $\varepsilon \oplus \varepsilon < r$. Then,

$$B(x, \varepsilon, \tfrac{t}{2}) \cap B(y, \varepsilon, \tfrac{t}{2}) = \emptyset.$$

Indeed, if the intersection contains some point (z), then

$$\mu(x, z, t) \leq \mu(x, z, \tfrac{t}{2}) \oplus \mu(z, y, \tfrac{t}{2}) \leq \varepsilon \oplus \varepsilon < r.$$

The obtained contradiction completes the proof. □

Theorem 2. *Let $(X, \mu, \oplus)$ be an RGV-fuzzy metric space and τ be the induced topology. Let $\{x_n\}$ be a sequence in X. Then, $\lim_{n\to\infty} x_n = x_0$, if and only if $\lim_{n\to\infty} \mu(x_n, x_0, t) = 0$ for any $t > 0$.*

Proof. Assume that $\lim_{n\to\infty} x_n = x_0$ and take some $t > 0$. Further, let $0 < r < 1$; then, there exists n_0 such that $x_n \in B(x, r, t)$ for all $n > n_0$, and hence, $\mu(x_n, x_0, t) < r$. However, this means that $\lim_{n\to+\infty} \mu(x_n, x_0, t) = 0$.

Conversely, if for each $t > 0$ $\lim_{n\to+\infty} \mu(x_n, x_0, t) = 0$, for $0 < r < 1$, $n_0 \in \mathbb{N}$ exists such that $\mu(x_n, x_0, t) < r$ for all $n \geq n_0$. Thus, $x_n \in B(x_0, r, t)$ for all $n \geq n_0$ and hence, $\lim_{n\to\infty} x_n = x_0$. □

Given an RGV-fuzzy metric space $(X, \mu, \oplus)$, we define the closed ball with center $x \in X$, radius $0 < r < 1$, and at the level $t > 0$ by

$$B[x, r, t] = \{y \in X : \mu(x, y, t) \leq r\}.$$

Theorem 3. *A closed ball is a closed set in the induced topology.*

Proof. Let $y \in cl(B[x, r, t])$, where cl denotes the closure operator in the induced topology. Since X is first countable, the sequence $\{x_n\}$ in $B[x, r, t]$ exists such that $\lim_{n\to\infty} x_n = y$ and hence, $\lim_{n\to\infty} \mu(x_n, y, t) = 0$.

We fix some $\varepsilon > 0$. Then,

$$\mu(x, y, t + \varepsilon) \leq \mu(x, x_n, t) \oplus \mu(x_n, y, \varepsilon).$$

Now, we take the limits of both sides of the above inequality when $n \to \infty$ and, referring to the continuity of the t-conorm $\oplus$ and continuity of the function μ, we get

$$\mu(x, y, t + \varepsilon) \leq \lim_{n\to\infty} \mu(x, x_n, t) \oplus \lim_{n\to\infty} \mu(x_n, y_n, \varepsilon) \leq r \oplus 0 = r.$$

Since this is true for any $\varepsilon > 0$ and μ is continuous in the third argument, it follows that $\mu(x, y, t) \leq r$ and hence, $y \in B[x, r, t]$. Therefore, $B[x, r, t]$ is a closed set. □

From here, and noticing that each point in $B[x, r, t]$ can be reached as a limit of a sequence of points lying in $B(x, r, t)$, we get the following corollary.

Corollary 2. *A closed ball ($B[x, r, t]$) is the closure of the open ball ($B(x, r, t)$).*

4. Conclusions

In this short note, we have defined an RGV-fuzzy metric as the dual version of a GV-fuzzy metric and illustrated how basic concepts and results of the theory of GV-fuzzy metric spaces can be reformulated and reproved in the terms of RGV-fuzzy metric spaces. It was not our aim to convince the reader that RGV-fuzzy metrics are "better" than GV-fuzzy metrics. Moreover, we believe that some authors will argue with us by confirming that GV-fuzzy metrics better correspond to the "spirit" of mathematics of fuzzy sets. Our aim here was to present an alternative approach to the theory of GV-fuzzy metrics and to illustrate through by some examples of statements and proofs in theory of RGV-fuzzy metric spaces. In our opinion, these proofs are more visible and easy than their original counterparts done in the terms of GV-fuzzy metric spaces. Besides, we believe that our revised interpretation of fuzzy metrics will be more convenient in some applications. Below, we outline two directions where in our opinion this approach could be helpful.

One of the fields where RGV-metrics could be helpful is in image restoration. As mentioned in Remark 1, a possible interpretation of the value $\mu(x, y, t)$ is that it reflects our ability to distinguish the distance between pixels x and y under some condition (say distance of observation or time) defined by the parameter t. By using the revised fuzzy metrics, the researcher will be provided with visual and easy to manage information about the influence of parameter t (e.g., distance or time) in the process of image restoration of a picture. This information could be useful, in particular, to find the value of parameter t at which the result of this process will be optimal.

In ref. [7], Bets and Šostak used so-called fragmentary GV-fuzzy pseudometrics in order to characterize the distance between two infinite words. This information is important for analyzing the analytic structure of the set of infinite words—the actual problem in combinatorics on words. Unfortunately, the visual interpretation of the obtained characterization looks rather bulky and is inconvenient to manage. Continuing this research, in our next paper (in preparation), we use revised GV-fuzzy metrics. The information described in terms of RGV-fuzzy metrics is coherent with our intuition and is more appropriate to manage. In particular, in case of revised fuzzy metrics, we have the natural interpretation of the standard situation: the longer the segments of two infinite words taken into consideration, the more precise the obtained information about the closeness of the two words is.

5. Some Additional Remarks

After this paper had been written, the author found that a t-conorm in the research related to fuzzy metrics was used by Park in [8] and some of his followers. Therefore, we feel it appropriate to compare the role of t-conorms in [8] and in our paper. Park in his paper introduced the concept of an *intuitionistic* (in Atannasov's sense [9]) fuzzy metric, and naturally, in the definition of an intuitionistic fuzzy metric on a set X he used *two functions* $M, N : X \times X \times (0, +\infty) \to (0, 1]$, satisfying inequality $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X, t > 0$. The first one of these functions, $M(x, y, t)$, describes the degree of *nearness*, while $N(x, y, t)$ describes the degree of *non-nearness* of points x, y on the level t. So, actually, M in Park's definition is *an ordinary* GV-*fuzzy metric* and therefore, it is based on the use of a t-norm $*$. On the other hand, function N, which in some sense complements function M, is based on a t-conorm $\Diamond$ (that is, probably, unrelated to the t-norm $*$). In contrast to the case of an intuitionistic fuzzy metric, we, when defining an RGV-fuzzy metric, started with a "classic" GV-metric and just reformulated the axioms from [3] by using involution. So, in our approach, a t-conorm $\oplus$ in the definition of a fuzzy metric is used to evaluate the degree of nearness of two points, and hence, it is opposite to the role of a t-conorm in the definition of an intuitionistic fuzzy metric.

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