On Hirata-Azumaya Galois Extensions

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Abstract

Let B be a Hirata separable and an Azumaya Galois extension of B^G with Galois group G . Then several characterizations of such a B are shown, and an equivalent condition is given for such a B satisfying the fundamental theory.

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1 Introduction

Let B be a ring with 1, G a finite automorphism group of B , C the center of B , and $B^G = \{b \in B \mid g(b) = b \text{ for each } g \in G\}$. In [3], [6], [9], the class of central Galois extensions B over C with Galois group G was studied. In [5], [8], the class of Hirata separable and Galois extensions B of B^G with Galois group G was investigated. In [1], [2], [14], the class of Galois extensions B of B^G with Galois group G such that B^G is an Azumaya C^G -algebra was explored, such a B is called an Azumaya Galois extension. The purpose of the present paper is to study the class of Hirata separable and Azumaya Galois extensions B of B^G with Galois group G . We shall show several characterizations of such a Galois extension B , and give an equivalent condition for such a B satisfying the fundamental theory which generalizes Theorem 3.4 in [10]. Examples are also

constructed to show that the class of Hirata separable and Azumaya Galois extensions is a broader class than the class of central Galois extensions, and is properly contained in the class of Azumaya Galois extensions and the class of Hirata separable Galois extensions.

2 Basic Definitions and Notations

Let B be a ring with 1, C the center of B , G a finite automorphism group of B , B^G the set of elements in B fixed under each element in G , and A a subring of B with the same identity 1. We call B a separable extension of A if there exist $\{a_i, b_i \text{ in } B, i = 1, 2, \dots, m \text{ for some integer } m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A , and an Azumaya algebra is a separable extension of its center ([4]). A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B -bimodule ([8]).

We call B a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i \text{ in } B, i = 1, 2, \dots, m\}$ for some integer m such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for each $g \in G$ ([3]). A ring B is called a Galois algebra over R if B is a Galois extension of R such that R is contained in the center C of B , and a central Galois algebra over R if $R = C$ ([6]). A ring B is called a commutator Galois extension of B^G with Galois group G if $V_B(B^G) (= \{b \in B \mid ab = ba \text{ for each } a \in A\})$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$ ([13]). We call B is a Hirata separable Galois extension of B^G with Galois group G if it is a Hirata separable and Galois extension of B^G with Galois group G ([8]). A Galois extension B with Galois group G is called an Azumaya Galois extension of B^G if B^G is an Azumaya C^G -algebra ([1], [2]). If B is a Hirata separable and an Azumaya Galois extension of B^G with Galois group G , we call B a Hirata-Azumaya Galois extension of B^G with Galois group G , which we are going to study in the present paper. A Galois extension B of B^G with Galois group G is called satisfying the fundamental theory if $\alpha : H \longrightarrow B^H$ is a one-to-one correspondence between the set of subgroups of G and the set of separable extensions A of B^G in B with the inverse map $\alpha^{-1} : A \longrightarrow G(A)$ where $G(A) = \{g \in G \mid g(a) = a \text{ for all } a \in A\}$.

Throughout this paper, we assume that B is a Galois extension of B^G with Galois group G , C the center of B , $J_g = \{b \in B \mid bx = g(x)b \text{ for each } x \in B\}$ for a $g \in G$, and for a subring A of B with the same identity 1, $G(A) = \{g \in G \mid g(a) = a \text{ for all } a \in A\}$ and $V_B(A)$ the commutator (also called centralizer) subring of A in B .

3 Main Results

Keeping the definitions and notations in section 2, in this section, we shall show several characterizations of an Azumaya Galois extension B which is also a Hirata separable extension of B^G , and give an equivalent condition for such a B satisfying the fundamental theory. We begin with a very useful property of an Azumaya algebra given by S. Ikehata.

Lemma 3.1 ([5], Theorem 1) *Let B be an Azumaya algebra and A a subalgebra of B . If B is projective as a left A -module, then B is a Hirata separable extension of A .*

Theorem 3.2 *Let B be a Galois extension of B^G with Galois group G . Then the following statements are equivalent:*

- (1) B is a Hirata separable and an Azumaya Galois extension of B^G with Galois group G .
- (2) B is an Azumaya Galois extension of B^G with Galois group G such that $C = C^G$.
- (3) $B = B^G V_B(B^G)$ which is an Azumaya C^G -algebra and the order of G is invertible in B .
- (4) B is Azumaya and $V_B(B^G)$ is a central Galois algebra over C with Galois group $G|_{V_B(B^G)} \cong G$.

Proof. (1) $\implies$ (2) By hypothesis, B is an Azumaya Galois extension of B^G with Galois group G . Moreover, since B is a Hirata separable Galois extension of B^G , $V_B(V_B(B^G)) = B^G$ ([8], Proposition 4-(1)); and so $C \subset B^G$. Hence $C = C^G$.

(2) $\implies$ (1) Since B is a Galois extension of B^G with Galois group G , B is a separable extension of B^G and a left projective B^G -module. Also, since B is an Azumaya Galois extension of B^G with Galois group G , B^G is an Azumaya C^G -algebra. Hence B is a separable C^G -algebra by the transitivity property of separable extensions. Noting that $C = C^G$ by hypothesis, we conclude that B is an Azumaya C^G -algebra. Therefore, B^G is a subalgebra of the Azumaya C^G -algebra B such that B is left projective over B^G . Hence B is a Hirata separable extension of B^G by Lemma 3.1.

(2) $\implies$ (3) Since B is an Azumaya Galois extension of B^G with Galois group G , $B = B^G V_B(B^G)$ ([14], Theorem 3.1-(3)). Moreover, by (2) $\implies$ (1), B is an Azumaya C^G -algebra. Therefore it suffices to show that the order of G is invertible in B . In fact, by (2) $\implies$ (1), B is a Hirata separable Galois extension of B^G with Galois group G . Also, since B^G is a separable subalgebra of the Azumaya algebra B , $V_B(B^G)$ is a separable subalgebra of the Azumaya algebra B by the commutator theorem for Azumaya algebras ([4], Theorem 4.3, page 57). Hence the order of G is invertible in B ([8], Proposition 4-(3)).

(3) $\implies$ (2) Since B is a Galois extension of B^G with Galois group G , B is left projective over B^G . By hypothesis, B is an Azumaya C^G -algebra and B^G is a subalgebra of B . Hence B is a Hirata separable extension of B^G by Lemma 3.1, and so B is a Hirata separable Galois extension of B^G with Galois group G . Therefore, $V_B(V_B(B^G)) = B^G$ ([8], Proposition 4-(1)). Moreover, since the order of G is invertible in B , $V_B(B^G)$ is a separable subalgebra of the Azumaya algebra B ([8], Proposition 4-(3)). Thus $B^G (= V_B(V_B(B^G)))$ is also a separable subalgebra of the Azumaya algebra B over C^G by the commutator theorem for Azumaya algebras ([4], Theorem 4.3, page 57). Noting that $B = B^G V_B(B^G)$, we have that the center of B^G is C^G . Hence B^G is an Azumaya C^G -algebra; and so B is an Azumaya Galois extension of B^G with Galois group G .

(2) $\implies$ (4) Since B is an Azumaya Galois extension of B^G with Galois group G such that $C = C^G$, B is Azumaya and $V_B(B^G)$ is a Galois algebra over C with Galois group $G|_{V_B(B^G)} \cong G$ ([1], Theorem 2-(3)). Hence B is a commutator Galois extension of B^G with Galois group G ; and so C is the center of $V_B(B^G)$ ([13], Lemma 3.2). Therefore $V_B(B^G)$ is a central Galois algebra over C with Galois group $G|_{V_B(B^G)} \cong G$.

(4) $\implies$ (2) Since $V_B(B^G)$ is a central Galois algebra over C with Galois group $G|_{V_B(B^G)} \cong G$, B is a commutator Galois extension of B^G with Galois group G and $C = (V_B(B^G))^G \subset B^G$, and so $C = C^G$. Moreover, by hypothesis, B is an Azumaya C -algebra. Hence B^G is an Azumaya C^G -algebra ([12], Theorem 3.1). Therefore B is an Azumaya Galois extension of B^G with Galois group G such that $C = C^G$. This completes the proof.

Next, we show an equivalent condition for a Galois extension B as given in Theorem 3.2 satisfying the fundamental theory. We begin with two important lemmas given by T. Kanzaki.

Lemma 3.3 ([6], Proposition 1) *Let B be a Galois extension of B^G with Galois group G . Then $V_B(B^G) = \oplus \sum_{g \in G} J_g$.*

Lemma 3.4 ([7], Proposition 3.1) *Let B be a Galois extension of a separable R -algebra with Galois group G . Then B^H is a separable R -algebra for every subgroup H of G .*

Corollary 3.5 *Let B be an Azumaya Galois extension of B^G with Galois group G . Then B^H is a separable C^G -algebra for every subgroup H of G .*

Theorem 3.6 *Let B be a Galois extension of B^G with Galois group G as given in Theorem 3.2. Then B satisfies the fundamental theorem if and only if for any separable extension A of B^G in B , $V_B(A) = \oplus \sum_{g \in G(A)} J_g$.*

Proof. ($\implies$) Let A be a separable extension of B^G in B . Since B satisfies the fundamental theorem, we have that $A = B^{G(A)}$; and so $V_B(A) = V_B(B^{G(A)})$. But B is a Galois extension of $B^{G(A)}$ with Galois group $G(A)$, so $V_B(B^{G(A)}) = \oplus \sum_{g \in G(A)} J_g$ by Lemma 3.3. Hence $V_B(A) = V_B(B^{G(A)}) = \oplus \sum_{g \in G(A)} J_g$.

($\impliedby$) Since B is a Hirata separable Galois extension of B^G with Galois group G , $H \longrightarrow B^H$ is a one-to-one map from the set of subgroups of G to the set of subextensions A of B^G in B ([11], Corollary 3.5). Since B is an Azumaya Galois extension of B^G with Galois group G , B^H is a separable C^G -algebra by Corollary 3.5; and so a separable extension of B^G in B for every subgroup H of G . Hence $H \longrightarrow B^H$ is a one-to-one map from the set of subgroups of G to the set of separable extensions A of B^G in B . Therefore it suffices to show that for any separable extension A of B^G in B , $A = B^{G(A)}$. In fact, since A is a separable extension of B^G and B^G is a separable C -algebra by Theorem 3.2-(2), A is a separable subalgebra of the Azumaya algebra B over C . Hence $A = V_B(V_B(A))$ by the commutator theorem for Azumaya algebras ([4], Theorem 4.3, page 57). Moreover, since B is an Azumaya Galois extension of B^G with Galois group G , for a subgroup $G(A)$ of G , $B^{G(A)}$ is separable over C^G by Corollary 3.5, so $B^{G(A)}$ is a separable subalgebra of the Azumaya algebra B over C . Therefore $B^{G(A)} = V_B(V_B(B^{G(A)}))$ by the commutator theorem for Azumaya algebras. Now, since B is a Galois extension of $B^{G(A)}$ with Galois group $G(A)$, $V_B(B^{G(A)}) = \oplus \sum_{g \in G(A)} J_g$ by Lemma 3.3, and by hypothesis, $V_B(A) = \oplus \sum_{g \in G(A)} J_g$. Thus $V_B(A) = \oplus \sum_{g \in G(A)} J_g = V_B(B^{G(A)})$. Hence $A = V_B(V_B(A)) = V_B(V_B(B^{G(A)})) = B^{G(A)}$. This completes the proof.

We remark that since a central Galois algebra B over C with Galois group G is a Hirata separable and an Azumaya Galois extension of B^G with Galois group G , Theorem 3.6 generalizes the characterization for a central Galois algebra satisfying the fundamental theorem given by G. Szeto and L. Xue.

Corollary 3.7 ([10], Theorem 3.4.) *Let B be a central Galois algebra over C with Galois group G . Then B satisfies the fundamental theorem if and only if for any separable subalgebra A of B , $V_B(A) = \oplus \sum_{g \in G(A)} J_g$.*

We conclude the present paper with three examples of a Galois extension B with Galois group G to illustrate that (1) B is an Azumaya Galois extension of B^G with Galois group G , but B is not a Galois extension with Galois group G as given in Theorem 3.2; (2) B is a Hirata separable Galois extension of B^G with Galois group G , but B is not a Galois extension with Galois group G as given in Theorem 3.2; (3) B is a Galois extension with Galois group G as given in Theorem 3.2, but not a central Galois algebra with Galois group G .

Example 3.8 Let $M_2(\mathbb{Z})$ be the ring of 2×2 matrices over the integers $\mathbb{Z}$, $B = M_2(\mathbb{Z}) \times M_2(\mathbb{Z})$ the direct product of two copies of $M_2(\mathbb{Z})$, and $G = \{1, g \mid g(x, y) = (y, x) \text{ for } (x, y) \in B\}$. Then

- (1) G is an automorphism group of B of order 2, and $B^G = \{(x, x) \mid x \in M_2(\mathbb{Z})\} \cong M_2(\mathbb{Z})$;
- (2) Let C be the center of B . Then $C \cong \mathbb{Z} \times \mathbb{Z}$ and $C^G = \{(x, x) \mid x \in \mathbb{Z}\} \cong \mathbb{Z}$;
- (3) B is a Galois extension of B^G with Galois group G and a Galois system

$$\{a_1 = (I_2, 0_2), a_2 = (0_2, I_2); \quad b_1 = a_1, b_2 = a_2\}$$

where I_2 is the identity matrix in $M_2(\mathbb{Z})$ and 0_2 is the zero matrix in $M_2(\mathbb{Z})$; that is $a_1 b_1 + a_2 b_2 = (I_2, I_2)$ and $a_1 g(b_1) + a_2 g(b_2) = (0_2, 0_2)$;

- (4) B^G is an Azumaya C^G -algebra;
- (5) By (3) and (4), B is an Azumaya Galois extension of B^G with Galois group G ;
- (6) Since 2, the order of G , is not invertible in B , B is not a Galois extension of B^G with Galois group G as given in Theorem 3.2 by Theorem 3.2-(3).

Example 3.9 Let $\mathbb{R}[i, j, k]$ be the quaternion algebra over the real field $\mathbb{R}$, A a noncommutative ring with center $\mathbb{R}$, but not an Azumaya $\mathbb{R}$ -algebra, $B = A \otimes_{\mathbb{R}} \mathbb{R}[i, j, k]$, and $G = \{1, g_i, g_j, g_k\}$ where $g_i(a \otimes x) = a \otimes i x i^{-1}$, $g_j(a \otimes x) = a \otimes j x j^{-1}$, and $g_k(a \otimes x) = a \otimes k x k^{-1}$ for all $a \in A$ and $x \in \mathbb{R}[i, j, k]$. Then

- (1) It is easy to check that the center of B is $\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R} \cong \mathbb{R}$ and $B^G = A \otimes_{\mathbb{R}} \mathbb{R} \cong A$;
- (2) B is a Galois extension with Galois group G with a Galois system $\{1 \otimes 1, 1 \otimes i, 1 \otimes j, 1 \otimes k; \frac{1}{4} \otimes 1, -\frac{1}{4} \otimes i, -\frac{1}{4} \otimes j, -\frac{1}{4} \otimes k\}$;
- (3) Since G is inner induced by the elements $\{1 \otimes 1, 1 \otimes i, 1 \otimes j, 1 \otimes k\}$, B is a Hirata separable extension of B^G ([8], Corollary 3);
- (4) By (2) and (3) B is a Hirata separable Galois extension of B^G with Galois group G ;
- (5) Since $B^G = A \otimes_{\mathbb{R}} \mathbb{R} \cong A$ which is not an Azumaya algebra over its center $\mathbb{R}$, B is not an Azumaya Galois extension with Galois group G .

Example 3.10 Let $M_2(\mathbb{R})$ be the ring of 2×2 matrices over the real numbers $\mathbb{R}$, $B = M_2(\mathbb{R})[i, j, k]$ the quaternion ring of quaternions i, j and k over $M_2(\mathbb{R})$, and $G =$ the inner automorphism group of B induced by $\{1, i, j, k\}$. Then

- (1) B is a Galois extension of B^G with Galois group G and has a Galois system $\{a_1 = \frac{1}{2}I_2, a_2 = \frac{i}{2}I_2, a_3 = \frac{j}{2}I_2, a_4 = \frac{k}{2}I_2; b_1 = a_1, b_2 = -a_2, b_3 = -a_3, b_4 = -a_4\}$ where I_2 is the identity matrix in $M_2(\mathbb{R})$;

(2) The center C of B is $\mathbb{R}I_2$ such that $C^G = C$ and $B^G (= M_2(\mathbb{R}))$ is an Azumaya C -algebra;

(3) By (1) and (2), B is an Azumaya Galois extension of B^G with Galois group G such that $C^G = C$, that is, B is a Galois extension with Galois group G as given in Theorem 3.2, by Theorem 3.2-(2);

(4) Since $B^G = M_2(\mathbb{R})$ which is not the center of B , B is not a central Galois algebra with Galois group G .

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