Limitations of the Describing Function for Limit Cycle Prediction

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***Abstract*—We consider comparator-based nonlinear feedback systems, and use Tsypkin's method to develop a strategy with which to find systems with low-pass linear part for which the describing function technique erroneously predicts limit cycles. We produce an infinite set of examples of systems with very low-pass linear parts for which the describing function technique predicts spurious limit cycles, and also provide a more practical example in which limit cycles are erroneously predicted.**

***Index Terms*—Describing functions, limit cycles, Tsypkin's method.**

I. Introduction

Consider a nonlinear feedback circuit which consists of a comparator and a linear element (see Fig. 1). One way to check for limit cycles is to use the describing function method. This is an approximate method—it is generally assumed to work well when the system under consideration has low-pass linear part, $G(s)$. We produce examples of systems that are as low pass as one could desire but for which the describing function method predicts spurious limit cycles. We make use of a form of the Tsypkin locus analysis in order to determine the types of systems for which the describing function technique does not work properly.

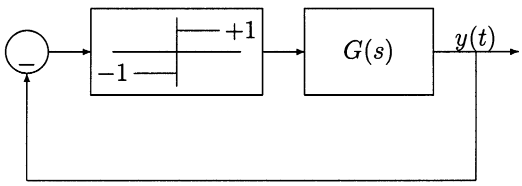

Fig. 1. Block diagram of a nonlinear oscillator.

Let $G(j\omega)$ be the frequency response of the linear part of the system and let $D(M)$ be the describing function of the nonlinear element. We show that when the nonlinear elements is a comparator then if $G(j\omega)$ is tangent, or nearly tangent, to $-1/D(M)$, then the describing function technique erroneously predicts a limit cycle, and we show how to quantify the term "nearly tangent." We provide two infinite sets of examples for which the describing function erroneously predicts limit cycles. We also consider a more practical example in detail.

The general result that we describe—that the near tangency of the graphs of $G(j\omega)$ and $-1/D(M)$ can lead the describing function technique to erroneously predict limit cycles—has been remarked upon previously [3], [5, p. 186]. We *prove* that the phenomenon exists in the case of a comparator nonlinearity, and we show how to *quantify* the phrase "nearly tangent" using exact methods.

II. The Describing Function Analysis

The describing function of a comparator is $D(M) = 4/(\pi M)$. For a limit cycle to occur in the circuit of Fig. 1, the describing function technique requires a frequency for which $-G(j\omega)D(M) = 1$—a frequency for which the total gain "seen" by the limit cycle is one. This condition can be formulated as

$$G(j\omega) = -\frac{\pi M}{4}.$$

Graphically, this condition can be expressed as the necessity of an intersection of the graph of the frequency response and of the negative reciprocal of the describing function for some $\omega > 0$, $M > 0$. We note that in the case at hand, the graph of $-1/D(M)$, $M > 0$ is the negative-real axis.

We see that for a limit cycle to exist, the describing function method requires a positive frequency such that $G(j\omega)$ is real and negative. One cannot conclude that this is a requirement for the existence of a limit cycle; the describing function analysis is only approximate. It is generally used for systems whose linear part has at least a double zero at infinity—for systems that are rather low pass.

In deriving the describing function technique, one assumes that the output of the system is well approximated by a pure sine wave. Thus, the output of the comparator—whose input is just the inverted output—is a square wave that is positive in one half period and negative in the next [4, pp. 586–588]. In Section III, we see that in such cases there is an exact method for determining whether or not limit cycles exist.

We make use of the exact method to show that there exist an infinite set of examples with as many zeros at infinity as one desires for which the describing function method predicts the existence of a limit cycle when no limit cycle exists.

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III. Tsypkin Locus Analysis

When considering systems in which the nonlinear element is a comparator, one need not rely on the describing function method. Hamel [2] and Tsypkin [6] independently introduced an exact method for finding limit cycles when the nonlinearity is a comparator, and when the output of the comparator is a square wave that is positive in one half cycle and negative in the next. Making use of their ideas, we are able to present a necessary condition for the existence of oscillations; this condition is just one part of the condition that they derive, yet it is all that we need. Defining $\lambda(\omega)$ as

$$\lambda(\omega) \equiv -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\mathrm{Im}(G((2n-1)j\omega)}{2n-1}$$

we find that if there is to be a limit cycle with frequency $\tilde{\omega}$ and if the output of the circuit is continuous, (which is the case if $G(j\omega)$ has at least one zero at infinity), then $\lambda(\tilde{\omega}) = 0$ at that frequency. Our condition can easily be derived from the material found in [1, pp. 431–436].

IV. Criteria for Failure of the Describing Function Method

From the Tsypkin locus analysis, we see that the describing function method does not lead to the correct condition for oscillations in a system whose nonlinear part is a comparator. Rather than the condition for limit cycles being that the value of the transfer function must be negative at one point, the correct condition is that a certain weighted sum of values of the imaginary part of the transfer function evaluated at various frequencies must be zero. If the system is "sufficiently" low pass, we find that the sum which must be zero reduces to

$$\lambda(\omega) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\mathrm{Im}(G((2n-1)j\omega)}{2n-1} \approx -\frac{4}{\pi}\mathrm{Im}(G(j\omega)).$$

This leaves us with the requirement that

$$\mathrm{Im}(G(j\omega)) = 0$$

which is a weaker form of the condition to which the describing function method leads us.

We now consider cases in which the low-pass condition is true, but for which the describing function method still does not work. There is one set of simple examples—examples in which the linear part's imaginary part is always of one sign save at the point $\tilde{\omega}$ at which the imaginary part is zero and the real part is negative. In such cases, the graphs of $G(j\omega)$ and $-1/D(M)$ are tangent, and the describing function method predicts limit cycles at $\tilde{\omega}$. However, as $\mathrm{Im}(G(j\omega))$ always has the same sign when $\omega > 0$, $\omega \neq \tilde{\omega}$, we find that for all $\omega > 0$, $\lambda(\omega) \neq 0$; the Tsypkin locus analysis shows that such a system cannot support a limit cycle.

V. The Low-Pass System

As a rule, when ω increases the value of a system's frequency response, $G(j\omega)$, wanders continuously from quadrant to quadrant with the imaginary part changing its sign from time to time. The simplest examples of systems for which the describing function technique fails are systems in which the linear part of the system consists of cascaded subsystems whose phase is either constant, or whose phase always has the same sign. The simplest useful examples of subsystems of the first type are subsystems of the form $1/s^n$ or s^m. The simplest example of subsystems of the second type are phase-lag and phase-lead controllers.

We are led to consider systems of the form

$$G(s) = \frac{1}{s^{3+4n}}\left(\frac{\sqrt{3}s+1}{\frac{s}{\sqrt{3}}+1}\right)^3, \qquad n \geq 0.$$

For $s = j\omega, \omega > 0$, the first term has a fixed phase of $-(270+360n)^\circ$. The term

$$\frac{\sqrt{3}s+1}{\frac{s}{\sqrt{3}}+1}$$

is a phase-lead compensator. It is well known (see, for example, [4]) that such a component's phase is always positive, that its phase reaches its maximum at $\omega = 1$, and that at this point the phase is $+30^\circ$. After raising this term to the third power, we find that the term always contributes a positive phase, we find that $G(j\cdot 1) < 0$, and we find that for positive ω, $\mathrm{Im}(G(j\omega)) \leq 0$.

According to the describing function method, there is a limit cycle with angular frequency $\omega = 1$. Because of the fact that $\mathrm{Im}(G(j\omega)) \leq 0$ for all $\omega \neq 1$, we know that $\lambda(\omega) \neq 0$; the Tsypkin locus analysis shows that this system *does not* support limit cycles. Though for large n this system is as low pass as one could want, the describing function method erroneously predicts limit cycles.

We note that a second set of similar examples is provided by systems of the form:

$$G(s) = \frac{1}{s^{1+4n}}\left(\frac{\frac{s}{\sqrt{3}}+1}{\sqrt{3}s+1}\right)^3, \qquad n \geq 0.$$

Here too, the describing function method erroneously predicts the existence of limit cycles at $\omega = 1$. Even if the phase of $G(j\omega)$ is less than -180° for a "little while," the system still should not support a limit cycle. In Section VI, we show how to quantify this notion.

VI. A More Practical System

The low-pass systems previously described provide an infinite set of cautionary examples related to the describing function. We now consider a more practical example. Suppose that

$$G(s) = \frac{1}{s(s+1)}\left(\frac{10s+1}{(28+12\sqrt{5})s+1}\right)^2.$$

It is easy to verify that the imaginary part of this function is equal to zero at exactly one point and is otherwise always negative (see Fig. 2). Thus, this provides another example of a system for which the describing function mistakenly predicts a limit cycle. Additionally, the first part of the transfer function could be the transfer function of a dc motor and the second is just a twice-repeated phase-lag compensator. This is a reasonable system to consider, and even here, the describing function fails to work properly.

One need not require that the frequency response $G(j\omega)$ be tangent to the real axis in order to find that the describing function technique does not work properly. Consider the system described by

$$G(s) = \frac{1}{s(s+1)}\left(\frac{10s+1}{as+1}\right)^2.$$

Using a simple computer program, it is easy to calculate $\lambda(\omega)$ for various values of a and ω. Using such a program, we have found that when the linear part of the system is $G(s)$, the system does not support limit cycles if $a \leq 56.5$ though it does support limit cycles if $a \geq 57.0$. (Note that $28 + 12\sqrt{5} = 54.833$). We have quantified the expression "nearly tangent" in our example, and the method used can be used whenever the nonlinear element is a comparator.

Recall that the graph of the negative reciprocal of the describing function is the entire negative-real axis. To demonstrate how far from tangency the two graphs can be and how negative the value of the imaginary part can get without causing limit cycles to occur, consider the partial Nyquist plot of the above system with $a = 56.5$ given in Fig. 3. We find that the imaginary part comes close to $+0.15$.

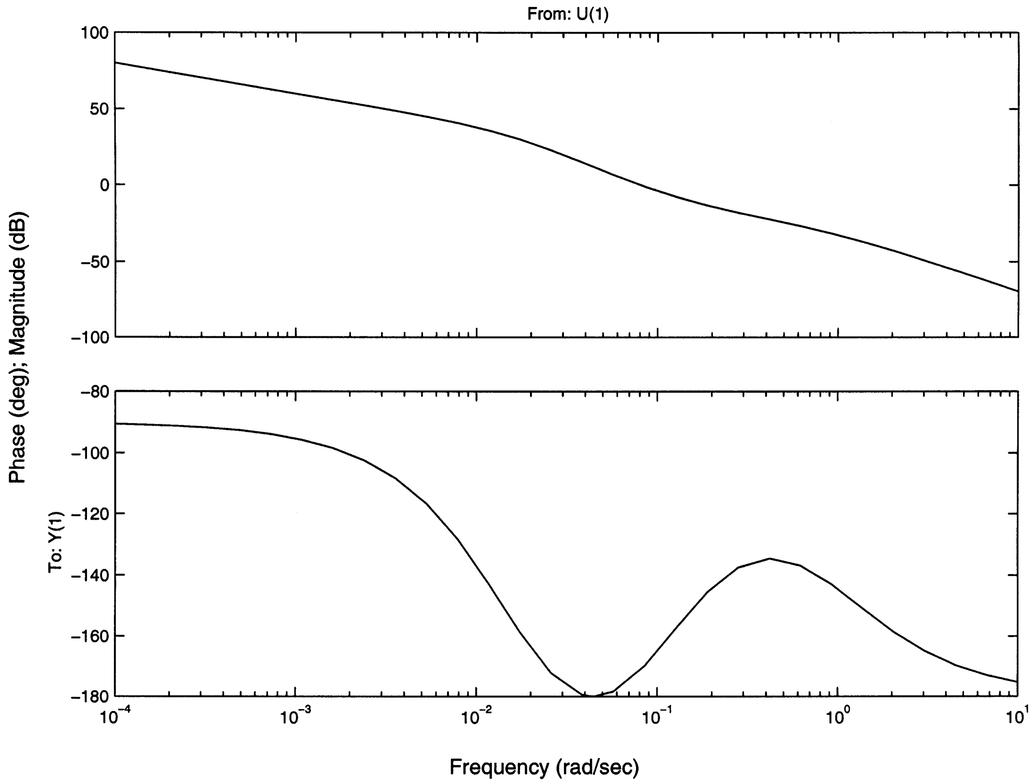

Fig. 2. The bode plots.

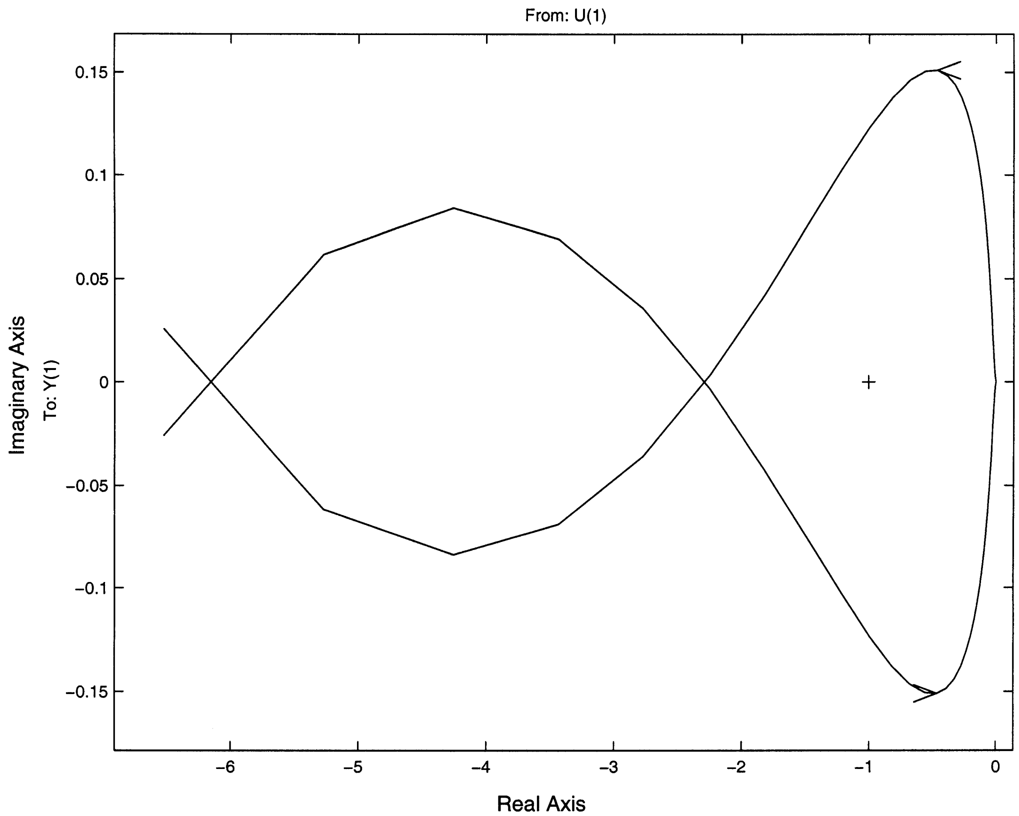

Fig. 3. The Nyquist plot.

VII. Conclusion

The describing function technique is an approximate technique for determining the existence of limit cycles in systems with a single nonlinearity. It is generally considered to be a reasonably accurate technique provided that the linear part of the system on which the analysis is being performed is low pass. It is known to, at times, erroneously predict limit cycles if the plots of the frequency response and of the negative reciprocal of the describing function are "nearly tangent." We have produced an infinite set of examples of systems that are as low pass as one could desire but for which we *prove* that the describing function erroneously predicts the existence of limit cycles. We have shown by example how one can use the Tsypkin locus analysis to *quantify* how "nearly tangent" the graphs of the transfer function and the negative reciprocal of the describing function must be in order for the describing function technique to erroneously predict limit cycles.

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