A NOTE ON SOME THEOREMS OF R. DATKO

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Abstract:

The asymptotic behavior of the evolution families is a widely interesting topic in mathematics over time. In 1930, O. Perron was the first one who established the connection between the asymptotic behavior of the solution of the homogenous differential equation and the associated non-homogeneous equation, in finite dimensional spaces. Further, the result was extended for infinite dimensional spaces. The case of dynamical systems described by evolution processes was studied by C. Chicone and Y. Latushkin. One of the most remarkable results in the theory of stability of dynamical systems has been obtained by R. Datko in 1970 for the particular case of C_0 -semigroups. Practically, R. Datko defines a characterization for uniform exponential stability of the C_0 -semigroups. Later, it was proved that a similar characterization is also valid for two-parameter evolution families.

In this paper, we obtain different versions of a well-known theorem of R. Datko for uniform and nonuniform exponential bounded evolution families. More precisely, we obtain theorems that characterize the nonuniform and uniform exponential stability of evolution families with uniform and nonuniform exponential growth. We show that, if we choose K dependent of t_0 in the form of Datko's theorem used by C. Stoica and M. Megan, we obtain a result of nonuniform exponential stability, which is no longer possible in the original form of Datko's theorem.

In conclusion, we generalize the results initially obtained by Datko (1972) and Preda and Megan (1985), by presenting some sufficient conditions for the nonuniform exponential stability of evolution families with nonuniform exponential growth.

UDC Classification: 517.9; **DOI:** <http://dx.doi.org/10.12955/cbup.v5.1069>

Keywords: Evolution family, Nonuniform exponential stability, Datko method

Introduction

In the present paper we study the asymptotic behavior of evolution families. As a starting point for a vast amount of literature concerning this subject, we mention the pioneering work of O. Perron (Perron, 1930), who was the first who establish the connection between the asymptotic behavior of the solution of the differential equation

$$(A) \quad \dot{x}(t) = A(t)x(t) \quad (1)$$

and the associated non-homogeneous equation

$$(A, f) \quad \dot{x}(t) = A(t)x(t) + f(t) \quad (2)$$

in finite dimensional spaces, where A is a $n \times n$ dimensional, continuous and bounded matrix and f is a continuous and bounded function on $\mathbb{R}_+$. This idea was later developed by W. A. Coppel (Coppel, 1978) and P. Hartman (Hartman, 1964) for differential systems in finite dimensional spaces.

In (Massera & Schäffer, 1958) and (Massera & Schäffer, 1966), J. L. Massera and J. J. Schäffer study the same problem as O. Perron for differential systems in infinite dimensional spaces and prove that if the pairs (L^1, L^∞) and (L^p, L^∞) , $p > 1$, are admissible to (A) , then it is a uniform exponential dichotomic differential system.

Further developments for differential systems in infinite dimensional spaces can be found in the monograph of J. L. Daleckij and M. G. Krein (Daleckij & Krein, 1974). The case of dynamical systems described by evolution processes was studied by C. Chicone, Y. Latushkin (Chicone & Latushkin, 1999).

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One of the most remarkable results in the theory of stability of dynamical systems has been obtained by Datko (Datko, 1970) in 1970 for the particular case of C_0 -semigroups. Thus in (Datko, 1970) it was established that all the trajectories $T(\cdot)x$ (of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$) have an exponential decay as $t \rightarrow \infty$ (i.e. $\{T(t)\}_{t \geq 0}$ is uniformly exponentially stable) if and only if, for all vectors $x \in X$, the function $t \rightarrow \|T(t)x\|$ lies in $L^2(\mathbb{R}_+)$. Later, A. Pazy shows in (Pazy, 1972) and (Pazy, 1983) that the result remains valid if we replace $L^2(\mathbb{R}_+)$ with $L^p(\mathbb{R}_+)$, where $p \in [1, \infty)$.

In 1972, R. Datko extends in (Datko, 1972) the above result for two-parameter evolution families stating that an evolution family $\{\Phi(t, t_0)\}_{t \geq t_0 \geq 0}$ (with uniform exponential growth) is uniformly exponentially stable (i.e. there exist $N, \nu > 0$ such that $\|\Phi(t, t_0)\| \leq Ne^{-\nu(t-t_0)}$, for all $t \geq t_0 \geq 0$) if and only if there exists $p \in (0, \infty)$ such that $\sup_{t \geq 0} \int_t^\infty \|\Phi(\tau, t)x\|^p d\tau < \infty$, for all $x \in X$. It is worth noting that the above Datko's theorem already appears in (Daleckij & Krein, 1974) for the special case of differential systems.

A similar characterization for uniform exponential stability of evolution families was obtained by S. Rolewicz in 1986, in (Rolewicz, 1986). More precisely, the author proved that if, for a continuous non-decreasing mapping $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, with the properties $F(0) = 0$ and $F(t) > 0$ for all $t > 0$, and for an evolution family $\{\Phi(t, t_0)\}_{t \geq t_0 \geq 0}$ on a Banach space X , with exponential growth, the following relation holds $\sup_{t \geq 0} \int_t^\infty F(\|\Phi(\tau, t)x\|) d\tau < \infty$, for all $x \in X$, then the evolution family is uniformly exponentially stable.

Datko's result was extended to dichotomy by Preda and Megan (Preda & Megan, 1985) in 1985. We also mention the contributions of M. Megan, A. L. Sasu and B. Sasu, who in (Megan & Sasu, 2002) and (Megan & Sasu, 2003) obtained generalizations of some results of Datko, Rolewicz and van Neerven. Other generalizations of Datko's theorem for asymptotic stability of evolution families were obtained by C. Buşe in (Buşe, 1994) and (Buşe, 1997).

Generalizations of the above Datko's theorem for skew-evolution semiflows appear in (Stoica & Megan, 2010). Also, in this paper, Datko's theorem is used in an equivalent form, i.e. an evolution family with uniform exponential growth is uniform exponentially stable if and only if there exist $p \in (0, \infty)$ and $k > 0$ such that $(\int_t^\infty \|\Phi(\tau, t)x\|^p d\tau)^{1/p} \leq k\|x\|$, for all $t \geq 0$ and $x \in X$.

In 2010, L. Barreira and C. Valls introduced in (Barreira & Valls, 2010) some appropriate adapted norms (which can be seen as Lyapunov norms), to show an equivalence between the admissibility of their associated L^p spaces ($p \in [1, \infty]$) and the nonuniform exponential stability of certain evolution families. The result is extended by the same authors in 2011 (Barreira & Valls, 2011) to the case of nonuniform exponential dichotomy where they also establish a collection of admissible Banach spaces for any given nonuniform exponential contraction.

The aim of our paper is to obtain theorems that characterize the uniform and nonuniform exponential stability of evolution families with nonuniform and uniform exponential growth using the Lyapunov norms introduced by L. Barreira and C. Valls in (Barreira & Valls, 2010) and Datko's method. We show that, if we choose K dependent of t_0 in the form of Datko's theorem used by C. Stoica and M. Megan in (Stoica & Megan, 2010), we obtain a result of nonuniform exponential stability, which is no longer possible in the original form of Datko's theorem.

Preliminaries

Let X be a Banach space and $\mathbb{B}(X)$ the space of all linear and bounded operators acting on X . The norms on X and on $\mathbb{B}(X)$ will be denoted by $\|\cdot\|$.

Definition 1: An evolution family $\{\Phi(t, t_0)\}_{t \geq t_0}$ on $\mathbb{R}_+$ is a family of operators $\Phi(t, t_0) \in \mathbb{B}(X)$, $t \geq t_0 \geq 0$, satisfying:

- (i) $\Phi(t, t) = I$, for all $t \in \mathbb{R}_+$, where I denotes the identity on X ;
- (ii) $\Phi(t, s)\Phi(s, t_0) = \Phi(t, t_0)$, for all $t \geq s \geq t_0 \geq 0$;
- (iii) the map $\Phi(\cdot, t_0)x$ is continuous on $[t_0, \infty)$ for all $x \in X$ and $\Phi(t, \cdot)x$ is continuous on $[0, t]$ for all $x \in X$.

If there exist $M, \omega > 0$ such that

$$\|\Phi(t, t_0)x\| \leq M e^{\omega(t-t_0)} \|x\|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X,$$

then it is said that the evolution family $\{\Phi(t, t_0)\}_{t \geq t_0}$ has uniform exponential growth.

If there exist $M : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ and $\omega > 0$ such that

$$\|\Phi(t, t_0)x\| \leq M(t_0) e^{\omega(t-t_0)} \|x\|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X,$$

then it is said that the evolution family $\{\Phi(t, t_0)\}_{t \geq t_0}$ has nonuniform exponential growth.

We denote $\|x\|_{t_0} = \sup_{t \geq t_0} e^{-\omega(t-t_0)} \|\Phi(t, t_0)x\|$. It is easy to see that $\|\cdot\|_{t_0}$ defines a norm on X , for all $t_0 \geq 0$, and

$$\|x\| \leq \|x\|_{t_0} \leq M(t_0) \|x\|, \text{ for all } t_0 \geq 0.$$

If $\{\Phi(t, t_0)\}_{t \geq t_0}$ has a uniform exponential growth, then $\|\cdot\|_{t_0}$ is uniformly equivalent (with respect to t_0) to the norm $\|\cdot\|$.

Remark 1: If $\{\Phi(t, t_0)\}_{t \geq t_0}$ has uniform exponential growth, then

$$\|\Phi(t, t_0)x\|_t \leq e^{\omega(t-t_0)} \|x\|_{t_0}, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X.$$

Indeed we can see that for all $t \geq t_0 \geq 0$ we have that

$$\begin{aligned} \|\Phi(t, t_0)x\|_t &= \sup_{\tau \geq t} e^{-\omega(\tau-t)} \|\Phi(\tau, t)\Phi(t, t_0)x\| = \\ &= \sup_{\tau \geq t} e^{-\omega(\tau-t_0-t+t_0)} \|\Phi(\tau, t_0)x\| = e^{\omega(t-t_0)} \sup_{\tau \geq t} e^{-\omega(\tau-t_0)} \|\Phi(\tau, t_0)x\| \leq \\ &\leq e^{\omega(t-t_0)} \|x\|_{t_0}, \text{ for all } x \in X. \end{aligned}$$

We can also see that for all $t_0 \geq 0$ and $x \in X$ the function $[t_0, \infty) \ni s \rightarrow \|\Phi(s, t_0)x\|_s \in \mathbb{R}_+$ is lower semi-continuous, therefore it is measurable.

Main results

In what follows we will present some sufficient conditions for nonuniform exponential stability of evolution families with nonuniform exponential growth in terms of Datko's theory, which was also highlighted by C. Stoica and M. Megan in (Stoica & Megan, 2010).

Theorem 1: Let Φ be an evolution family with nonuniform exponential growth. Then there exist $K : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$, $p > 0$ such that

$$\left(\int_t^\infty \|\Phi(\tau, t_0)x\|_\tau^p d\tau \right)^{1/p} \leq K(t_0) \|\Phi(t, t_0)x\|_t, \text{ for all } t \geq t_0 \text{ and } x \in X$$

if and only if there exist $N, \nu : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that

$$\|\Phi(t, t_0)x\|_t \leq N(t_0) e^{-\nu(t_0)(t-t_0)} \|x\|_{t_0}, \text{ for all } t \geq t_0 \geq 0.$$

Proof. Necessity. Let $x \in X$, $t \geq t_0 + 1$ and $\varphi : [t_0 + 1, \infty) \rightarrow \mathbb{R}_+^*$, $\varphi(t) = \int_{t-1}^\infty \|\Phi(\tau, t_0)x\|_\tau^p d\tau$. It is obvious that

$$\varphi(\tau) \leq -K^p(t_0) \varphi(\tau) \text{ for all } \tau \geq t_0 + 1.$$

Therefore, by integrating on the interval $[t_0 + 1, t]$ with respect to τ ,

$$\frac{1}{K^p(t_0)}(t - t_0 - 1) \leq \ln \frac{\varphi(t_0 + 1)}{\varphi(t)} \text{ for all } t \geq t_0 + 1.$$

It follows that

$$\varphi(t) e^{\frac{1}{K^p(t_0)}(t-t_0-1)} \leq \varphi(t_0 + 1) \leq -K^p(t_0)\dot{\varphi}(t_0 + 1) = K^p(t_0)\|x\|_{t_0}^p.$$

The last relation is equivalent to

$$\int_{t-1}^{\infty} \|\Phi(\tau, t_0)x\|_{\tau}^p d\tau \leq e^{-\frac{1}{K^p(t_0)}(t-t_0-1)} K^p(t_0)\|x\|_{t_0}^p, \text{ for all } t \geq t_0 + 1 \text{ and } x \in X. \quad (3)$$

Let $t \geq t_0 + 1$, $\tau \in [t - 1, t]$ and $x \in X$.

We have that $\|\Phi(t, t_0)x\|_t \leq e^{\omega} \|\Phi(\tau, t_0)x\|_{\tau}$. We integrate the last relation on $[t - 1, t]$ with respect to τ and by (3) it follows that

$$\begin{aligned} \|\Phi(t, t_0)x\|_t^p &\leq e^{\omega p} \int_{t-1}^t \|\Phi(\tau, t_0)x\|_{\tau}^p d\tau \leq \\ &\leq e^{\omega p} \int_{t-1}^{\infty} \|\Phi(\tau, t_0)x\|_{\tau}^p d\tau \leq e^{\omega p} e^{-\frac{1}{K^p(t_0)}(t-1-t_0)} K^p(t_0)\|x\|_{t_0}^p. \end{aligned}$$

Therefore

$$\|\Phi(t, t_0)x\|_t \leq e^{\omega} K(t_0) e^{\frac{1}{pK^p(t_0)}(t-t_0)} e^{-\frac{1}{pK^p(t_0)}(t-t_0)} \|x\|_{t_0}, \text{ for all } t \geq t_0 + 1 \text{ and } x \in X. \quad (4)$$

Let $t \in [t_0, t_0 + 1)$ and $x \in X$. In this case

$$\|\Phi(t, t_0)x\|_t \leq e^{\omega} e^{\frac{1}{pK^p(t_0)}(t-t_0)} e^{-\frac{1}{pK^p(t_0)}(t-t_0)} \|x\|_{t_0} \leq e^{\omega + \frac{1}{pK^p(t_0)}} e^{-\frac{1}{pK^p(t_0)}(t-t_0)} \|x\|_{t_0}. \quad (5)$$

We denote $N(t_0) = e^{\omega + \frac{1}{pK^p(t_0)}} \max\{1, K(t_0)\}$ and $\nu(t_0) = \frac{1}{pK^p(t_0)}$. By relations (4) and (5) we can conclude that

$$\|\Phi(t, t_0)x\|_t \leq N(t_0) e^{-\nu(t_0)(t-t_0)} \|x\|_{t_0}, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X.$$

Sufficiency. Only some simple calculations are needed in order to draw the conclusion. □

Remark 2: The above result can be considered as the stronger version of the main result of the article (Stoica & Megan, 2010).

This is proved by the following corollary.

Corollary 1: Let $\{\Phi(t, t_0)\}_{t \geq t_0}$ be an evolution family with uniform growth. If there exist $K : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ and $p > 0$ such that

$$\left(\int_t^{\infty} \|\Phi(\tau, t_0)x\|_{\tau}^p d\tau \right)^{1/p} \leq K(t_0) \|\Phi(t, t_0)x\|, \text{ for all } t \geq t_0 \text{ and } x \in X,$$

then there exist $N, \nu : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that

$$\|\Phi(t, t_0)x\| \leq N(t_0) e^{-\nu(t_0)(t-t_0)} \|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Proof. It follows from Theorem 1. □

Remark 3: If $\sup_{t_0 \geq 0} K(t_0) = K < \infty$, for all $t_0 \geq 0$, in the corollary above, then, there exists $p > 0$ such that

$$\left(\int_t^\infty \|\Phi(\tau, t_0)x\|^p d\tau \right)^{1/p} \leq K \|\Phi(t, t_0)x\|, \text{ for all } t \geq t_0 \text{ and } x \in X,$$

if and only if there exist $N, \nu > 0$ such that

$$\|\Phi(t, t_0)x\| \leq N e^{-\nu(t-t_0)} \|x\|, \text{ for all } t \geq t_0 \text{ and } x \in X.$$

Proof. Sufficiency. If $\tau \geq t \geq t_0$, then

$$\|\Phi(\tau, t_0)x\| \leq N e^{-\nu(\tau-t_0)} \|\Phi(t, t_0)x\|.$$

It follows that

$$\left(\int_t^\infty \|\Phi(\tau, t_0)x\|^p d\tau \right)^{1/p} \leq \frac{N}{(\nu p)^{1/p}} \|\Phi(t, t_0)x\|, \text{ for all } t \geq t_0 \geq 0 \text{ and } x \in X.$$

Necessity. It is obvious by Theorem 1. □

Remark 4: The converse of Corollary 1 is false.

Indeed, let $X = \mathbb{R}$ and $\Phi: \{(t, t_0) \in \mathbb{R}_+^2 : t \geq t_0\} \rightarrow \mathbb{R}$, $\Phi(t, t_0) = \frac{t_0^2 + 1}{t^2 + 1}$.

We have that

$$\int_t^\infty \frac{t_0^2 + 1}{\tau^2 + 1} d\tau = (t_0^2 + 1) \left(\frac{\pi}{2} - \arctan t \right) \leq \frac{\pi}{2} (t_0^2 + 1) = K(t_0). \quad (6)$$

We now assume that there exist $N, \nu: \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that $\|\Phi(t, t_0)\| \leq N(t_0) e^{-\nu(t_0)(t-t_0)}$, for all $t \geq t_0$. If $t_0 = 0$, then it follows that

$$\frac{1}{t^2 + 1} \leq N(0) e^{-\nu(0)t}, \text{ for all } t \geq 0.$$

Therefore

$$\frac{e^{\nu(0)t}}{t^2 + 1} \leq N(0), \text{ for all } t \geq 0.$$

If $t \rightarrow \infty$, the last relation is equivalent to $\infty \leq N(0)$, which is absurd.

Therefore we can conclude that there do not exist two functions $N, \nu: \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that $\|\Phi(t, t_0)\| \leq N(t_0) e^{-\nu(t_0)(t-t_0)}$, for all $t \geq t_0$. This shows that it is impossible to choose K independent of t_0 in (6).

Corollary 2: Let $\{\Phi(t, t_0)\}_{t \geq t_0 \geq 0}$ be an evolution family with uniform growth. The following statements are equivalent:

- (i) there exist $N, \nu > 0$ such that $\|\Phi(t, t_0)x\| \leq N e^{-\nu(t-t_0)} \|x\|$, for all $t \geq t_0 \geq 0$ and $x \in X$;
- (ii) there exist $k, p > 0$ such that $(\int_t^\infty \|\Phi(\tau, t)x\|^p d\tau)^{1/p} \leq k \|x\|$, for all $t \geq 0$ and $x \in X$;
- (iii) there exist $k, p > 0$ such that $(\int_t^\infty \|\Phi(\tau, t_0)x\|^p d\tau)^{1/p} \leq k \|\Phi(t, t_0)x\|$, for all $t \geq t_0 \geq 0$ and $x \in X$.

Proof. It follows from Remark 3 and Theorem 1. □

Conclusion

In this paper are obtained different types of R. Datko theorems which characterize the uniform and nonuniform exponential stability of evolution families with uniform and nonuniform exponential growth. Practically, the result initially obtained by Datko (1972) and Preda and Megan (1985) was extended using so-called Lyapunov norms (i.e.

$$\|x\|_{t_0} = \sup_{t \geq t_0} e^{-\omega(t-t_0)} \|\Phi(t, t_0)x\|, \forall t_0 \geq 0, \forall x \in X$$

introduced by L. Barreira and C. Valls in (Barreira & Valls, 2010).

We shown that, if we choose K dependent of t_0 in the form of Datko's theorem used by C. Stoica and M. Megan, we obtain a result of nonuniform exponential stability of the evolution family, which is no longer possible in the original form of Datko's theorem. So, considering Φ an evolution family with nonuniform exponential growth, then there exist $K : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$, $p > 0$ such that

$$\left(\int_t^\infty \|\Phi(\tau, t_0)x\|_\tau^p d\tau \right)^{1/p} \leq K(t_0) \|\Phi(t, t_0)x\|_t, \text{ for all } t \geq t_0 \text{ and } x \in X$$

if and only if there exist $N, \nu : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$ such that

$$\|\Phi(t, t_0)x\|_t \leq N(t_0) e^{-\nu(t_0)(t-t_0)} \|x\|_{t_0}, \text{ for all } t \geq t_0 \geq 0.$$

It was also proved, that the main result obtained in this paper (Theorem 1) is even stronger then the already existing results in the literature.

In conclusion, in this paper, we presented some new sufficient conditions for the nonuniform exponential stability of evolution families with nonuniform exponential growth in terms of Datko's theory.

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