# Downward Löwenheim-Skolem Theorem and Interpolation in Logics with Constructors

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## Abstract

The present paper describes a method for proving Downward Löwenheim-Skolem Theorem within an arbitrary institution satisfying certain logic properties. In order to demonstrate the applicability of the present approach, the abstract results are instantiated to many-sorted first-order logic and preorder algebra. In addition to the first technique for proving Downward Löwenheim-Skolem Theorem, another one is developed, in the spirit of institution-independent model theory, which consists of borrowing the result from a simpler institution across an institution comorphism. As a result the Downward Löwenheim-Skolem Property is exported from first-order logic to partial algebras, and from higher-order logic with intensional Henkin semantics to higher-order logic with extensional Henkin semantics. The second method successfully extends the domain of application of Downward Löwenheim-Skolem Theorem to other non-conventional logical systems for which the first technique may fail. One major application of Downward Löwenheim-Skolem Theorem is interpolation in constructor-based logics with universally quantified sentences. The interpolation property is established by borrowing it from a base institution for its constructor-based variant across an institution morphism. This result is important as interpolation for constructor-based first-order logics is still an open problem.
**Keywords:** Institution, algebraic specification, first-order logic, interpolation, constructor-based logic.

## 1 Introduction

The Downward Löwenheim-Skolem Theorem (abbreviated DLST) is a fundamental result in model theory which states that if a countable theory is consistent then it has a countable model. The framework adopted here is the theory of institutions [18] which is a category-based formalisation of the intuitive notion of logical system. Institutions constitute a meta-theory on logical systems similar to the manner in which universal algebra constitutes a meta-theory for groups and rings. The first proof of DLST within an arbitrary institution enjoying certain properties is due to [22]. The method used is that of forcing invented by Paul Cohen [8, 9], and introduced in institutional model theory in [25]. The approach is very general but is applicable only to countable languages with the semantics restricted to models that have non-empty carrier sets. These restrictions are no longer needed in the present study as a novel technique is employed for proving a more refined version of DLST: for any model there exists an elementary submodel of cardinality greater or equal than the cardinality of the set of formulas and less than the cardinality of the underlying model. [1] The categorical assumptions used here are easy to check in concrete logics, and for this reason the abstract theorems can be instantiated to many institutions, some of them explicitly described here, and others just mentioned.

There are examples of more refined institutions which cannot be cast in this abstract framework and for which we believe that the standard methods for proving DLST cannot be replicated. Therefore, in addition to the first technique for establishing Downward Löwenheim-Skolem Property (abbreviated DLSP), we develop another one, in the spirit of institution-independent model theory. Instead of developing directly the result within a given institution, one may borrow it from a simpler institution via an adequate encoding, expressed as an institution *comorphisms* [20]. More concretely, given an institution comorphism $I \to I'$ such that the institution $I'$ has DLSP then the institution $I$ can be established to have DLSP. We demonstrate the applicability of our *borrowing* result with two examples: we "export" the DLSP from first-order logic to partial algebra and from higher-order logic with intensional Henkin semantics to higher-order logic with extensional Henkin semantics.

DLST is used to borrow interpolation from a base institution for its constructor-based variant across forgetful *institution morphisms* [18]. Constructor-based institutions are obtained from a base institution by enhancing the syntax with a sub-signature of constructor operators and restricting the semantics to reachable models, which consist of constructor-generated elements. The sentences and the satisfaction condition are preserved from the base institution, while the signature morphisms are restricted such that the reducts of models that are reachable in the target signature are again reachable in the source signature. Several algebraic specification languages incorporate features to express reachability and to deal with constructors like, for instance, Larch [28], CASL [1] or CITP [27]. Given a constructor-based institution $I^c$ over a base institution $I$ there exists a natural forgetful institution morphism $I^c \to I$. The institution $I^c$ can be established to have the interpolation property if $I$ has the interpolation property and satisfies some extra conditions. In [21], an interpolation result is proved for constructor-based institutions with Horn sentences of the form $(\forall X) \wedge H \Rightarrow C$, where $H$ is a set of atomic formulas and $C$ is an atomic formula. In this paper, another interpolation result is established for constructor-based institutions, but in this case, the sentences consist of universally quantified formulas of the form $(\forall X)\rho$, where $\rho$ is a quantifier-free formula.

The paper is organised as follows. The first technical section introduces the institution-theoretic preliminaries and recalls the necessary fundamental concepts of institution-independent model theory such as internal logic, basic sets of sentences, reachable models, exactness, and elementary morphisms. In Section 3, we develop an institution-independent version of DLST that is applicable to many concrete institutions. Section 4 studies the translation of DLSP along institution comorphisms and illustrates its applicative power with examples which cannot be captured in the previous abstract setting. In Section 5, we borrow interpolation from a base institution for its constructor-based variant across forgetful institution morphisms. Section 6 concludes the paper and discusses the future work.

[1] Note that DLST is applicable when the cardinality of the underlying model is strictly greater than the cardinality of the set of sentences which is at least $\omega$.

# 2 Institutions

In this section we define the necessary model-theoretic infrastructure within the theory of institutions to prove our abstract results.

## 2.1 Preliminaries

We assume the reader is familiar with basic notions of category theory such as category, functor, pushout, natural transformation, etc., which are omitted here. With a few exceptions, we use the terminology and the notations from [30]. In this sense, we denote by $|\mathcal{C}|$ the collection of objects of a category $\mathcal{C}$, by $\mathcal{C}(A,B)$ the collection of arrows from $A$ to $B$, by $f;g$ the composition of arrows $f$ and $g$ in diagrammatic order (first apply $f$ then apply $g$), and by $1_A$ the identity arrow of an object $A$. Given a set $X$, the notation $|X|$ usually means the cardinality of $X$. In order to avoid confusion, we let $card(X)$ to denote the cardinality of $X$.

We say that $\mathcal{C}'$ is a *broad* subcategory of $\mathcal{C}$ if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ such that $|\mathcal{C}'| = |\mathcal{C}|$. We say that $\mathcal{C}'$ is a *full* subcategory of $\mathcal{C}$ if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ such that $\mathcal{C}(A,B) = \mathcal{C}'(A,B)$ for all objects $A,B \in |\mathcal{C}'|$. Given a category $\mathcal{C}$, the subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is *closed under pushouts* if for any span of arrows $A' \xleftarrow{g} A \xrightarrow{f} A_1$ such that $f \in \mathcal{C}'$ there exists a pushout $\{A' \xleftarrow{g} A \xrightarrow{f} A_1, A' \xrightarrow{f'} A'_1 \xleftarrow{g_1} A_1\}$ such that $f' \in \mathcal{C}'$. The subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is *strongly closed under pushouts* if it is closed under pushouts and for any pushout $\{A' \xleftarrow{g} A \xrightarrow{f} A_1, A' \xrightarrow{f'} A'_1 \xleftarrow{g_1} A_1\}$ such that $f \in \mathcal{C}'$ we have $f' \in \mathcal{C}'$.

Assume a functor $F : \mathcal{C} \to \mathcal{D}$. For any subcategory $\mathcal{C}' \subseteq \mathcal{C}$ the *restriction* $F|_{\mathcal{C}'} : \mathcal{C}' \to \mathcal{D}$ of $F$ to $\mathcal{C}'$ is defined the same as $F$ on the objects and arrows of $\mathcal{C}'$, i.e. $F|_{\mathcal{C}'}(A) = F(A)$ for all objects $A \in |\mathcal{C}'|$ and $F|_{\mathcal{C}'}(f) = F(f)$ for all arrows $f \in \mathcal{C}'$. For any subcategory $\mathcal{D}' \subseteq \mathcal{D}$ such that $F(\mathcal{C}) \subseteq \mathcal{D}'$ the *corestriction* $F|^{\mathcal{D}'} : \mathcal{C} \to \mathcal{D}'$ of $F$ to $\mathcal{D}'$ is defined the same as $F$ on the objects and arrows of $\mathcal{C}$. When there is no danger of confusion we may denote both the restriction and corestriction simply by $F$. A functor $F : \mathcal{C} \to \mathcal{D}$ *lifts* $(\mathcal{C}_1, \mathcal{C}_2)$*-pushouts*, where $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{C}$, if for any pushout $\{F(A') \xleftarrow{F(g)} F(A) \xrightarrow{F(f)} F(A_1), F(A') \xrightarrow{u'} B'_1 \xleftarrow{v_1} F(A_1)\}$ in $\mathcal{D}$ such that $A \xrightarrow{f} A_1 \in \mathcal{C}_1$ and $A \xrightarrow{g} A' \in \mathcal{C}_2$ there exists a pushout $\{A' \xleftarrow{g} A \xrightarrow{f} A_1, A' \xrightarrow{f'} A'_1 \xleftarrow{g_1} A_1\}$ such that $F(f') = u'$ and $F(g_1) = v_1$.

Our study is based on naive set theory for which we assume the *axiom of choice*. Given a finite set of symbols $S$, a *string* over $S$ is any finite sequence of symbols from $S$. The set of all strings over $S$ is denoted by $S^*$, and the empty sequence is denoted by $\varepsilon$. We also define $S^+$ to be the set of nonempty strings $S^* - \{\varepsilon\}$.

## 2.2 Definition and Examples

The concept of institution formalises the intuitive notion of logical system, and has been defined by Goguen and Burstall in the seminal paper [18].

**Definition 1** An *institution* $I = (\mathbb{S}ig^I, \mathbb{S}en^I, \mathbb{M}od^I, \models^I)$ consists of

(1) a category $\mathbb{S}ig^I$, whose objects are called *signatures*,

(2) a functor $\mathbb{S}en^I : \mathbb{S}ig^I \to \mathbb{S}et$, providing for each signature $\Sigma$ a set whose elements are called *($\Sigma$-)sentences*,

(3) a functor $\mathbb{M}od^I : (\mathbb{S}ig^I)^{op} \to \mathbb{CAT}$, providing for each signature $\Sigma$ a category whose objects are called *($\Sigma$-)models* and whose arrows are called *($\Sigma$-)morphisms*,

(4) a relation $\models^I_\Sigma \subseteq |\mathbb{M}od^I(\Sigma)| \times \mathbb{S}en^I(\Sigma)$ for each signature $\Sigma \in |\mathbb{S}ig^I|$, called *($\Sigma$-)satisfaction*, such that for each morphism $\varphi : \Sigma \to \Sigma'$ in $\mathbb{S}ig^I$, the following *satisfaction condition* holds:

$$M' \models^I_{\Sigma'} \mathbb{S}en^I(\varphi)(e) \text{ iff } \mathbb{M}od^I(\varphi)(M') \models^I_\Sigma e$$

for all $M' \in |\mathbb{M}od^I(\Sigma')|$ and $e \in \mathbb{S}en^I(\Sigma)$.

When there is no danger of confusion, we omit the superscript from the notations of the institution components; for example $\mathbb{S}ig^I$ may be simply denoted by $\mathbb{S}ig$. We denote the *reduct* functor $\mathbb{M}od(\varphi)$ by $\_\upharpoonright_\varphi$ and the sentence translation $\mathbb{S}en(\varphi)$ by $\varphi(\_)$. When $M = M'\upharpoonright_\varphi$ we say that $M$ is the $\varphi$-*reduct* of $M'$ and $M'$ is a $\varphi$-*expansion* of $M$. A signature morphism $\varphi : \Sigma \to \Sigma'$ is *conservative* if all $\Sigma$-models have a $\varphi$-expansion. Given a signature $\Sigma$ and two sets of $\Sigma$-sentences $E_1$ and $E_2$, we write $E_1 \models\!\mid E_2$ whenever $E_1 \models E_2$ and $E_2 \models E_1$. A set of $\Sigma$-sentences is *consistent* if there exists a $\Sigma$-model satisfying it. A set of $\Sigma$-sentences $\Gamma$ is *maximal consistent* if it is consistent, and for any other consistent set of $\Sigma$-sentences $\Gamma'$ such that $\Gamma \subseteq \Gamma'$ we have $\Gamma = \Gamma'$.

**Example 2 (First-Order Logic (FOL) [18] )** The signatures are triplets $(S,F,P)$, where $S$ is the set of sorts, $F = (F_{w\to s})_{(w,s)\in S^*\times S}$ is the ($S^*\times S$ -indexed) set of operation symbols, and $P = (P_w)_{w\in S^*}$ is the ($S^*$-indexed) set of relation symbols. If $w = \varepsilon$, an element of $F_{w\to s}$ is called a *constant symbol*, or a *constant*. By a slight notational abuse, we let $F$ and $P$ also denote $\bigcup_{(w,s)\in S^*\times S} F_{w\to s}$ and $\bigcup_{w\in S^*} P_w$ respectively. A signature morphism between $(S,F,P)$ and $(S',F',P')$ is a triplet $\varphi = (\varphi^{st},\varphi^{op},\varphi^{rl})$, where $\varphi^{st} : S \to S'$, $\varphi^{op} : F \to F'$, $\varphi^{rl} : P \to P'$ such that for all $(w,s) \in S^* \times S$ we have $\varphi^{op}(F_{w\to s}) \subseteq F'_{\varphi^{st}(w)\to\varphi^{st}(s)}$, and for all $w \in S^*$ we have $\varphi^{rl}(P_w) \subseteq P'_{\varphi^{st}(w)}$. When there is no danger of confusion, we may let $\varphi$ denote each of $\varphi^{st}$, $\varphi^{op}$, $\varphi^{rl}$.

Given a signature $\Sigma = (S,F,P)$, a $\Sigma$-model is a triplet

$$M = ((M_s)_{s\in S}, (M^{w,s}_\sigma)_{(w,s)\in S^*\times S,\sigma\in F_{w\to s}}, (M^w_\pi)_{w\in S^*,\pi\in P_w})$$

interpreting each sort $s$ as a set $M_s$, each operation symbol $\sigma \in F_{w\to s}$ as a function $M^{w,s}_\sigma : M^w \to M_s$ (where $M^w$ stands for $M_{s_1} \times \ldots \times M_{s_n}$ if $w = s_1 \ldots s_n$), and each relation symbol $\pi \in P_w$ as a relation $M^w_\pi \subseteq M^w$. When there is no danger of confusion we may let $M_\sigma$ and $M_\pi$ denote $M^{w,s}_\sigma$ and $M^w_\pi$, respectively. Morphisms between models are the usual $\Sigma$-morphisms, i.e., $S$-sorted functions that preserve the structure. The $\Sigma$-algebra of terms is denoted by $T_\Sigma$.

The $\Sigma$-sentences are obtained from

- equality atoms $t_1 = t_2$, where $t_1,t_2 \in (T_\Sigma)_s$, $s \in S$, or
- relational atoms $\pi(t_1,\ldots,t_n)$, where $\pi \in P_{s_1\ldots s_n}$, $t_i \in (T_\Sigma)_{s_i}$ and $s_i \in S$ for all $i \in \{1,\ldots,n\}$,

by applying for a finite number of times Boolean connectives and quantification over finite sets of variables.

Satisfaction is the usual first-order satisfaction and is defined using the natural interpretations of ground terms $t$ as elements $M_t$ in models $M$. The definitions of functors $\mathbb{S}en$ and

$\mathbb{M}od$ on morphisms are the natural ones: for any signature morphism $\varphi : \Sigma \to \Sigma'$, $\mathbb{S}en(\varphi) : \mathbb{S}en(\Sigma) \to \mathbb{S}en(\Sigma')$ translates sentences symbol-wise, and $\mathbb{M}od(\varphi) : \mathbb{M}od(\Sigma') \to \mathbb{M}od(\Sigma)$ is the forgetful functor.

**Example 3 (First-Order Equational Logic (FOEQL))** This institution is obtained from **FOL** by restricting the syntax to signatures with no predicate symbols.

**Example 4 (Universal First-Order Logic (UnivFOL))** This institution is obtained from **FOL** by restricting the syntax to universal sentences of the form $(\forall X)\rho$, where $\rho$ is a quantifier-free formula.

**Example 5 (Constructor-based first-order logic (CFOL))** The **CFOL** signatures are of the form $(S,F,F^c,P)$, where $(S,F,P)$ is a first-order signature, and $F^c \subseteq F$ is a distinguished sub-family of sets of operation symbols called *constructors*. The constructors determine the set of *constrained* sorts $S^c \subseteq S$: $s \in S^c$ iff there exists a constructor $\sigma \in F^c_{w\to s}$. We call the sorts in $S^l = S - S^c$ *loose*. We let $F^{S^c}$ denote the family of operation symbols of constrained sorts, i.e. $F^{S^c}_{w\to s} = \begin{cases} F_{w\to s} & \text{if } s \in S^c \\ \emptyset & \text{if } s \in S^l \end{cases}$ for all $(w,s) \in S^* \times S$.

The $(S,F,F^c,P)$-sentences are the usual *first-order sentences*.

The $(S,F,F^c,P)$-models are the usual first-order structures $M$ with the carrier sets for the constrained sorts consisting of interpretations of terms formed with constructors and elements of loose sorts, i.e. there exists a set $C$ of constants of loose sorts and a function $f : C \to M$ such that for every constrained sort $s \in S^c$ the function $f^\#_s : (T_{(S,F^c)}(C))_s \to M_s$ is a surjection, where $f^\# : T_{(S,F^c)}(C) \to M\restriction_{(S,F^c)}$ is the unique extension of $f$ to a $(S,F^c)$-morphism.

A signature morphism $\varphi : (S,F,F^c,P) \to (S',F',F'^c,P')$ in **CFOL** is a first-order signature morphism $\varphi : (S,F,P) \to (S',F',P')$ such that the constructors are preserved along the signature morphisms (i.e. if $\sigma \in F^c$ then $\varphi(\sigma) \in F'^c$) and no "new" constructors are introduced for "old" constrained sorts (i.e. if $s \in S^c$ and $\sigma' \in F'^c_{w'\to\varphi(s)}$ then there exists $\sigma \in F^c_{w\to s}$ such that $\varphi(\sigma) = \sigma'$). Variants of **CFOL** were studied in [3] and [2].

**Example 6 (Preorder Algebra (POA) [14, 15])** The **POA** signatures are just ordinary algebraic signatures, i.e. **FOEQL** signatures. The **POA** models are *preordered algebras* which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e. monotonic) function. A *preordered algebra morphism* is just a family of preorder functors (preorder-preserving functions) which is also an algebra morphism.

The sentences have two kinds of atoms: equations and *preorder atoms*. A preorder atom $t \leq t'$ is satisfied by a preorder algebra $M$ when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \leq M_{t'}$ . Full sentences are constructed from equational and preorder atoms by applying Boolean connectives and first-order quantification.

**Example 7 (Partial Algebra (PA))** Here we consider the institution **PA** as employed by the specification language CASL [1]. Its signatures consist of tuples $(S,TF,PF)$, where $TF$ is a family of sets of total function symbols and $PF$ is a family of sets of partial function symbols such that $TF_{w\to s} \cap PF_{w\to s} = \emptyset$ for each arity $w$ and sort $s$. A signature morphism $\varphi : (S,TF,PF) \to (S',TF',PF')$ preserves total operation symbols: for all $(w,s) \in S^* \times S$ we have $\varphi(TF_{w\to s}) \subseteq TF'_{\varphi(w)\to\varphi(s)}$ and $\varphi(PF_{(w,s)}) \subseteq PF'_{\varphi(w)\to\varphi(s)} \cup TF'_{\varphi(w)\to\varphi(s)}$.

Models consist of algebras interpreting each total symbol in $TF$ as a total function and each partial symbol in $PF$ as a partial function. A *partial algebra morphism* $h : M \to N$ is a family of (total) functions $(M_s \stackrel{h_s}{\to} N_s)_{s \in S}$ indexed by the set of sorts $S$ such that $h_s(M_\sigma(m)) = N_\sigma(h_w(m))$ for each operation $\sigma : w \to s$ and each string of arguments $m \in M^w$ for which $M_\sigma(m)$ is defined.

We consider one kind of atomic sentences: *existence equality* $t \stackrel{e}{=} t'$. The existence equality $t \stackrel{e}{=} t'$ holds when both terms are defined and equal. The sentences are formed from these atomic sentences by applying Boolean connectives and quantification over finite sets of variables interpreted as total functions. The definedness predicate and strong equality can be introduced as notations: $def(t)$ stands for $t \stackrel{e}{=} t$, and $t \stackrel{s}{=} t'$ stands for $(t \stackrel{e}{=} t') \vee (\neg def(t) \wedge \neg def(t'))$.

**Example 8 (Higher-Order Logic with Henkin semantics (HNK))** **HNK** has been introduced and studied in [5] and [29]. In the present paper we consider a simplified version close to the "higher-order algebra" of [34] which does not consider λ-abstraction.

For any set $S$ of sorts, let $\overrightarrow{S}$ be the set of $S$-types defined as the least set such that $S \subseteq \overrightarrow{S}$ and $s_1 \to s_2 \in \overrightarrow{S}$ when $s_1, s_2 \in \overrightarrow{S}$. A **HNK** signature is a pair $(S,F)$, where $S$ is a set of sorts and $F$ is a family of sets of function symbols $F = (F_s)_{s \in \overrightarrow{S}}$. A signature morphism $\varphi : (S,F) \to (S',F')$ consists of a function $\varphi^{st} : S \to S'$ and a family of functions $(\varphi^{op}_s : F_s \to F'_{\varphi^{type}(s)})_{s \in \overrightarrow{S}}$ where $\varphi^{type} : \overrightarrow{S} \to \overrightarrow{S'}$ is the canonical extension of $\varphi^{st}$ to $\overrightarrow{S}$. For every signature $(S,F)$, a $(S,F)$-model $M$ interprets each sort $s \in S$ as a set, and each function symbol $\sigma \in F_s$ as an element of $M_s$, where for each types $s_1, s_2 \in \overrightarrow{S}$, $M_{s_1 \to s_2} \subseteq [M_{s_1} \to M_{s_2}] = \{f \text{ function} \mid f : M_{s_1} \to M_{s_2}\}$.

**Remark 9** Any **HNK** model satisfies the axiom $(\forall f, g)((\forall a) fa = ga) \Rightarrow (f = g)$.

A $(S,F)$-morphism $h : M \to N$ consists of a family of functions $(M_s \stackrel{h_s}{\to} N_s)_{s \in \overrightarrow{S}}$ such that $h(M_\sigma) = N_\sigma$ for all $\sigma \in F$, and the following diagram commutes

$$\begin{array}{ccc} M_{s_1} & \xrightarrow{f} & M_{s_2} \\ {\scriptstyle h_{s_1}} \downarrow & & \downarrow {\scriptstyle h_{s_2}} \\ N_{s_1} & \xrightarrow[h_{s_1 \to s_2}(f)]{} & N_{s_2} \end{array}$$

for all types $s_1, s_2 \in \overrightarrow{S}$ and functions $f \in M_{s_1 \to s_2}$.

Let $(S,F)$ be a **HNK** signature. The $\overrightarrow{S}$-sorted set of terms $T_{(S,F)}$ is defined as follows: for all $s \in \overrightarrow{S}$ and $\sigma \in F_s$ we have $\sigma \in (T_{(S,F)})_s$, and for all $s_1, s_2 \in \overrightarrow{S}$, $t \in (T_{(S,F)})_{s_1 \to s_2}$ and $t' \in (T_{(S,F)})_{s_1}$ we have $tt' \in (T_{(S,F)})_{s_2}$.[2] An $(S,F)$-equation is of the form $t_1 = t_2$, where $t_1$ and $t_2$ are terms of the same type. Sentences are constructed from equations by applying Boolean connectives and quantification over finite sets of variables of any type.

**Example 10 (Higher-Order Logic with intensional Henkin semantics (HNK$^i$))** **HNK$^i$** is a variation of **HNK** that has the same signatures and sentences as **HNK** but the models consist of intensional functions which are distinguished not only by their graph but also by a name. This means that a **HNK$^i$** model does not necessarily satisfy the axiom $(\forall f, g)((\forall a) fa = ga) \Rightarrow (f = g)$. It follows that for any **HNK** signature $(S,F)$, the category $\mathbb{M}od^{\mathbf{HNK}^i}(S,F)$ has

[2] Given a **HNK** signature $(S,F)$, the category $\mathbb{M}od^{\mathbf{HNK}}(S,F)$ does not have an initial object, in general [6].

an initial object given by the term model $T_{(S,F)}$, where for all $s_1, s_2 \in \overrightarrow{S}$ and $t \in (T_{(S,F)})_{s_1 \to s_2}$, the intensional function $t : (T_{(S,F)})_{s_1} \to (T_{(S,F)})_{s_2}$ is defined by $t(x) = tx$ for all $x \in (T_{(S,F)})_{s_1}$.

**Example 11 (Institution of presentations)** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution. A presentation $(\Sigma, E)$ consists of a signature $\Sigma \in |\mathbb{S}ig|$ and a set of sentences $E \subseteq \mathbb{S}en(\Sigma)$. A presentation morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $E' \models \varphi(E)$. The presentation morphisms form a category denoted $\mathbb{S}ig^{I^{pres}}$ with the composition inherited from the category of signatures. The model functor $\mathbb{M}od$ can be extended from the category of signatures $\mathbb{S}ig$ to the category of presentations $\mathbb{S}ig^{I^{pres}}$, by mapping a presentation $(\Sigma, E)$ to the *full subcategory* $\mathbb{M}od(\Sigma, E)$ of $\mathbb{M}od(\Sigma)$ consisting of models that satisfy $E$. The correctness of the definition of $\mathbb{M}od : \mathbb{S}ig^{I^{pres}} \to \mathbb{CAT}^{op}$ is guaranteed by the satisfaction condition for the base institution, i.e. for all $(\Sigma, E) \xrightarrow{\varphi} (\Sigma', E') \in \mathbb{S}ig^{I^{pres}}$ we have $M' \upharpoonright_{\varphi} \in |\mathbb{M}od(\Sigma, E)|$ for all $M' \in |\mathbb{M}od(\Sigma', E')|$. This leads to the *institution of presentations* $I^{pres} = (\mathbb{S}ig^{I^{pres}}, \mathbb{S}en, \mathbb{M}od, \models)$ over the base institution $I$, where the notations $\mathbb{S}en$ and $\models$ are overloaded such that

- for all $(\Sigma, E) \xrightarrow{\varphi} (\Sigma', E') \in \mathbb{S}ig^{I^{pres}}$ we have $\mathbb{S}en((\Sigma, E) \xrightarrow{\varphi} (\Sigma', E')) = \mathbb{S}en(\Sigma \xrightarrow{\varphi} \Sigma')$, and
- for all $M \in |\mathbb{M}od(\Sigma, E)|$ and $\rho \in \mathbb{S}en(\Sigma, E)$ we have $M \models_{(\Sigma,E)} \rho$ iff $M \models_{\Sigma} \rho$.

If $\mathcal{D} \subseteq \mathbb{S}ig$ is a subcategory of signature morphisms then we make the following notations:

(1) We let $\mathcal{D}^{pres}$ denote the subcategory of presentation morphisms $(\Sigma, E) \xrightarrow{\chi} (\Sigma', E')$ such that $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$.

(2) We overload the notation by letting $\mathcal{D}$ denote the subcategory of presentation morphisms $\chi : (\Sigma, E) \to (\Sigma', E')$ such that $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$ and $\chi(E) \models\!\mid E'$.

## 2.3 Internal Logic

The following institutional notions dealing with the logical connectives and quantifiers were defined in [36]. Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution and $\Sigma \in |\mathbb{S}ig|$ a signature.

(1) A $\Sigma$-sentence $\rho$ is a *semantic negation* of the $\Sigma$-sentence $e$ when for every $\Sigma$-model $M$ we have $M \models_{\Sigma} \rho$ iff $M \not\models_{\Sigma} e$.

(2) A $\Sigma$-sentence $\rho$ is a *semantic disjunction* of the finite set of $\Sigma$-sentences $E$ when for every $\Sigma$-model $M$ we have $M \models_{\Sigma} \rho$ iff $M \models_{\Sigma} e$ for some $e \in E$.

(3) A $\Sigma$-sentence $\rho$ is a *semantic existential* $\chi$*-quantification* of the $\Sigma'$-sentence $e'$, where $\chi : \Sigma \to \Sigma'$, when for every $\Sigma$-model $M$ we have $M \models_{\Sigma} \rho$ iff $M' \models_{\Sigma'} e'$ for some $\chi$-expansion $M'$ of $M$.

Distinguished negation $\neg\_$, disjunction $\vee\_$ and existential quantification $(\exists\chi)\_$ are called *first-order constructors* and they have the semantical meaning defined above.

**Assumption 12** Throughout this paper we assume the following commutativity property of the first-order constructors with the signature morphisms: for each signature morphism $\Sigma \xrightarrow{\varphi} \Sigma_1 \in \mathbb{S}ig$,

(1) $\varphi(\neg e) = \neg\varphi(e)$ for all $\Sigma$-sentences $\neg e$;

(2) $\varphi(\vee E) = \vee\varphi(E)$ for all $\Sigma$-sentences $\vee E$;

(3) for any $\Sigma$-sentence $(\exists\chi)e'$, where $\chi : \Sigma \to \Sigma'$, there exists
$$\begin{array}{ccc} \Sigma' & \overset{\varphi'}{\dashrightarrow} & \Sigma'_1 \\ {\scriptstyle\chi}\uparrow & \textit{pushout} & \uparrow{\scriptstyle\chi_1} \\ \Sigma & \underset{\varphi}{\longrightarrow} & \Sigma_1 \end{array}$$
such that $\varphi((\exists\chi)e') = (\exists\chi_1)\varphi'(e')$.

A variable for a **FOL** signature $\Sigma = (S,F,P)$ is a triple $(x,s,\Sigma)$, where $x$ is the name of the variable and $s \in S$ is the sort of the variable. Let $\chi : \Sigma \hookrightarrow \Sigma[X]$ be a signature extension with variables from $X$, where $X = (X_s)_{s\in S}$ is a $S$-sorted set of variables, $\Sigma[X] = (S, F \cup X, P)$ and for all $(w,s) \in S^* \times S$ we have
$$(F \cup X)_{w\to s} = \begin{cases} F_{w\to s} & \text{if } w \in S^+, \\ F_{w\to s} \cup X_s & \text{if } w = \varepsilon. \end{cases}$$
For any $\Sigma[X]$-sentence $\rho$, $(\exists X)\rho$ is an abbreviation for $(\exists\chi)\rho$. Consider a signature morphism $\varphi : \Sigma \to \Sigma_1$, where $\Sigma_1 = (S_1, F_1, P_1)$. Then $\varphi((\exists X)\rho) = (\exists X^{\varphi})\varphi'(\rho)$ where $X^{\varphi} = \{(x, \varphi^{st}(s), \Sigma_1) \mid (x,s,\Sigma) \in X\}$ and $\varphi' : \Sigma[X] \to \Sigma_1[X^{\varphi}]$ extends $\varphi$ canonically by mapping each variable $(x,s,\Sigma) \in X$ to $(x, \varphi^{st}(s), \Sigma_1) \in X^{\varphi}$. Note that
$$\begin{array}{ccc} \Sigma[X] & \overset{\varphi'}{\dashrightarrow} & \Sigma_1[X^{\varphi}] \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle\chi_1} \\ \Sigma & \underset{\varphi}{\longrightarrow} & \Sigma_1 \end{array}$$
is a pushout.

We assume another rather mild condition which can easily be checked in concrete example of institutions.

**Assumption 13** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\varphi : \Sigma \to \Sigma_1$ a signature morphism, $(\exists\chi)e'$ a $\Sigma$-sentence and $(\exists\chi_1)e'_1$ a $\Sigma_1$-sentence, where $\chi : \Sigma \to \Sigma'$ and $\chi_1 : \Sigma_1 \to \Sigma'_1$. If $\varphi((\exists\chi)e') = (\exists\chi_1)e'_1$ then there exists $\varphi' : \Sigma' \to \Sigma'_1$ such that
$$\begin{array}{ccc} \Sigma' & \overset{\varphi'}{\dashrightarrow} & \Sigma'_1 \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle\chi_1} \\ \Sigma & \underset{\varphi}{\longrightarrow} & \Sigma_1 \end{array}$$
is a pushout and $\varphi'(e') = e'_1$.

Very often quantification is considered only for a restricted class of signature morphisms. For example, quantification in **FOL** considers only the signature extensions with a finite number of variables. In this paper, a more general disjunction operator $\vee\_$ is considered, which is applicable to a (finite) set of sentences. Based on these constructors for sentences we can also define $\wedge$, *false*, $(\forall\chi)\_$ using the classical definitions. For example, $false = \vee\emptyset$.

**Definition 14** An institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ has

(1) negations if each sentence has a semantic negation,

(2) disjunctions if there exists a semantic disjunction of any finite set of sentences,

(3) $\mathcal{Q}$-existential quantification, where $\mathcal{Q} \subseteq \mathbb{S}ig$ is a broad subcategory of signature morphisms, if for each $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{Q}$ and $e' \in \mathbb{S}en(\Sigma')$ there exists a semantic existential $\chi$-quantification of $e'$.

## 2.4 Basic Sets of Sentences

A set of sentences $B \subseteq Sen(\Sigma)$ is *basic* [10] if there exists a $\Sigma$-model $M_B$ such that, for all $\Sigma$-models $M$, $M \models B$ iff there exists a morphism $M_B \to M$. We say that $M_B$ is a *basic model* of $B$. If in addition the morphism $M_B \to M$ is unique then the set $B$ is called *epi basic*.

**Lemma 15** Any set of atoms in **FOL** and **POA** is epi basic.

PROOF. Let $B$ be a set of atomic $(S,F,P)$-sentences in **FOL**. The basic model $M_B$ is the initial model of $B$ and it is constructed as follows: on the quotient $T_{(S,F)}/_{\equiv_B}$ of the term model $T_{(S,F)}$ by the congruence generated by the equational atoms of $B$, we interpret each relation symbol $\pi \in P$ by $(M_B)_\pi = \{(t_1/_{\equiv_B}, \ldots, t_n/_{\equiv_B}) \mid \pi(t_1, \ldots, t_n) \in B\}$. By defining an appropriate notion of congruence for **POA** models compatible with the preorder (see [13] or [7]) one may obtain the same result for **POA**. $\square$

The proof of Lemma 15 is well known, and it can be found, for example, in [10] or [12], but since we want to make use of the construction of the basic model, we include it for the convenience of the reader. In **PA** any set of ground existence equations is basic (see [7] for a proof of this fact). In **HNK**, a set of atomic sentences is not basic, in general [6].

## 2.5 Exactness

Institution theory is the only model theory that properly identified the exactness properties of logics [35] and then gradually realized their importance [16].

**Definition 16** An institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is

- *semi-exact* when the model functor $\mathbb{M}od : \mathbb{S}ig^{op} \to \mathbb{CAT}$ preserves pullbacks,
- *inductive-exact* when $\mathbb{M}od$ preserves inductive limits.
- *exact* when $\mathbb{M}od$ preserves limits.

Semi-exactness implies the following amalgamation property: for every pushout of signatures $\{\Sigma' \overset{\chi}{\leftarrow} \Sigma \overset{\varphi}{\rightarrow} \Sigma_1, \Sigma_1 \overset{\chi_1}{\rightarrow} \Sigma'_1 \overset{\varphi'}{\leftarrow} \Sigma'\}$ and each morphisms $h' \in \mathbb{M}od(\Sigma')$ and $h_1 \in \mathbb{M}od(\Sigma_1)$ such that $h'\restriction_\chi = h_1\restriction_\varphi$ there exists $h'_1 \in \mathbb{M}od(\Sigma'_1)$ such that $h'_1\restriction_{\varphi'} = h'$ and $h'_1\restriction_{\chi_1} = h_1$.

Inductive-exactness implies that for every limit ordinal $\lambda$, each inductive co-limit $(\Sigma_i \overset{v_{i,\lambda}}{\rightarrow} \Sigma_\lambda)_{i<\lambda}$ of an inductive diagram $(\Sigma_i \overset{v_{i,j}}{\rightarrow} \Sigma_j)_{i<j<\lambda}$ and any models $M_i \in |\mathbb{M}od(\Sigma_i)|$, where $i < \lambda$, we have: if $M_j\restriction_{v_{i,j}} = M_i$ for all $i < j < \lambda$ then there exists a model $M_\lambda \in |\mathbb{M}od(\Sigma_\lambda)|$ such that $M_\lambda\restriction_{v_{i,\lambda}} = M_i$ for all $i < \lambda$.

**Proposition 17** **FOL** is exact.

A proof of the above proposition can be found in [12].

## 2.6 Signature Extensions

In classical model theory, the models of interest are often constructed in an extension $\mathcal{L}_C$ of the initial language $\mathcal{L}$ with an infinite set of constants $C$. The following definitions give the categorical properties of the extension $\mathcal{L} \hookrightarrow \mathcal{L}_C$ that we need to obtain our results.

**Definition 18** For any institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, the power of a signature $\Sigma \in |\mathbb{S}ig|$ is $card(\mathbb{S}en(\Sigma))$.

**Definition 19 (Chains)** Let $\mathcal{C}$ be a category and $\lambda$ an ordinal.

(1) A $\lambda$-*chain* is a commutative diagram $(A_i \xrightarrow{v_{i,j}} A_j)_{i<j\leq\lambda}$, such that for each limit ordinal $\tau \leq \lambda$, $(v_{i,\tau})_{i<\tau}$ is the co-limit of $(v_{i,j})_{i<j<\tau}$. Note that the commutativity of the chain, which is implicit by functoriality, just means that $v_{i,j}; v_{j,k} = v_{i,k}$ for all $i < j < k \leq \lambda$.

(2) For any class of arrows $\mathcal{D} \subseteq \mathcal{C}$, a $(\lambda, \mathcal{D})$-*chain* is any $\lambda$-chain $(v_{i,j})_{i<j\leq\lambda}$ as above such that $v_{i,i+1} \in \mathcal{D}$ for each $i < \lambda$. We say that $A \xrightarrow{v} A_\lambda$ is a $(\lambda, \mathcal{D})$-chain (or simply, $\mathcal{D}$-*chain* when there is no need to make $\lambda$ explicit) if there exists a $(\lambda, \mathcal{D})$-chain $(A_i \xrightarrow{v_{i,j}} A_j)_{i<j\leq\lambda}$ such that $(A \xrightarrow{v} A_\lambda) = (A_0 \xrightarrow{v_{0,\lambda}} A_\lambda)$. We say that $\lambda$ is the *length* of $v$.

The definition of chains is used to formalise the DLSP in institutions. In **FOL**, we denote by $\mathcal{D}^{\mathbf{FOL}}$ the subcategory of signature extensions with a finite set of constants, and we show that any signature extension with constants can be regarded as a $\mathcal{D}^{\mathbf{FOL}}$-chain.

**Lemma 20** Let $\Sigma = (S, F, P)$ be a **FOL** signature, and $C$ a $S$-sorted set of constants different from the symbols in $\Sigma$. The inclusion $\Sigma \hookrightarrow \Sigma[C]$ is a $\mathcal{D}^{\mathbf{FOL}}$-chain, where $\Sigma[C]$ is obtained from $\Sigma$ by adding constants from $C$.

PROOF. Let $\lambda$ be the power of $\Sigma[C]$ and $\{\rho_i \in \mathbb{S}en(\Sigma[C]) \mid 0 < i < \lambda\}$ be an enumeration of the set $\mathbb{S}en(\Sigma[C])$. We construct a $(\lambda, \mathcal{D}^{\mathbf{FOL}})$-chain using the enumeration $\{\rho_i \in \mathbb{S}en(\Sigma[C]) \mid 0 < i < \lambda\}$. We define

(1) $\Sigma_0 = \Sigma$, and for any successor ordinal $i > 0$ let $C^i$ be the finite set of all constants from $C$ that occur in $\rho_i$ but not in $\Sigma_{i-1}$, $\Sigma_i = \Sigma_{i-1}[C^i]$, and $v_{i-1,i} = (\Sigma_{i-1} \hookrightarrow \Sigma_{i-1}[C^i])$.

(2) for all limit ordinals $\tau \leq \lambda$, let $C^\tau = \bigcup_{0<i<\tau} C^i$, $\Sigma_\tau = \Sigma[C^\tau]$ and $v_{i,\tau} = (\Sigma_i \hookrightarrow \Sigma[C^\tau])$ for all ordinals $i < \tau$.

Since for all successor ordinals $i < \lambda$ the set $C^i$ is finite, $(v_{i-1,i} : \Sigma_{i-1} \hookrightarrow \Sigma_{i-1}[C^i])$ is an arrow of $\mathcal{D}^{\mathbf{FOL}}$. For all limit ordinals $\tau \leq \lambda$, by the definition of $\Sigma_\tau$, we have that $(\Sigma_i \overset{v_{i,\tau}}{\hookrightarrow} \Sigma_\tau)_{i<\tau}$ is the colimit of $(\Sigma_i \overset{v_{i,j}}{\hookrightarrow} \Sigma_j)_{i<j<\tau}$. □

The following definitions provide conditions to prove an abstract version of DLST.

**Definition 21** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathcal{D}_\Sigma$ a class of $\mathcal{D}$-chains with a fixed domain $\Sigma \in |\mathbb{S}ig|$. Assume that $\lambda$ is the power of $\Sigma$. A $(\lambda, \mathcal{D})$-chain $\Sigma \xrightarrow{v} \Sigma_\lambda \in \mathcal{D}_\Sigma$ is a $\mathcal{D}_\Sigma$-*extension* of $\Sigma$ whenever

(1) the power of $\Sigma_\lambda$ is $\lambda$, and

(2) there exists an enumeration $\{\rho_i \in \mathbb{S}en(\Sigma_\lambda) \mid 0 < i < \lambda\}$ of $\mathbb{S}en(\Sigma_\lambda)$ such that for all successor ordinals $i$ with $0 < i < \lambda$ we have

   (a) $(\Sigma_{i-1} \xrightarrow{v_{i-1,i}} \Sigma_i) = (\Sigma_{i-1} \xrightarrow{u_i} \mathcal{L}_i); (\mathcal{L}_i \xrightarrow{w_i} \Sigma_i)$ for some $u_i, w_i \in \mathcal{D}$,

   (b) $\rho_i = v_{i,\lambda}(w_i(e_i))$ for some $e_i \in \mathbb{S}en(\mathcal{L}_i)$,

(c) if $e_i$ is an existentially quantified sentence, i.e. $e_i = (\exists\chi_i)\delta_i$ for some $(\mathcal{L}_i \xrightarrow{\chi_i} \Sigma'_i) \in \mathcal{D}$ and $\delta_i \in \mathbb{S}en(\Sigma'_i)$, and $(v_{0,i-1}; u_i; \chi_i) \in \mathcal{D}_\Sigma$ then there exists a conservative signature morphism $\psi_i : \Sigma'_i \to \Sigma_i$ such that $\chi_i; \psi_i = w_i$.

We show that in concrete examples of logical systems, signature extensions with an infinite set of constants for each sort satisfy the conditions of Definition 21.

**Lemma 22** Let $\Sigma = (S, F, P)$ be a first-order signature of power $\lambda$, and $C$ a $S$-sorted set of constants different from the symbols in $\Sigma$ such that $card(C_s) = \lambda$ for all sorts $s \in S$. The signature extension $v : \Sigma \hookrightarrow \Sigma[C]$ is a $\mathcal{D}^{\mathbf{FOL}}_\Sigma$-extension of $\Sigma$, where $\mathcal{D}^{\mathbf{FOL}}_\Sigma$ consists of all $\mathcal{D}^{\mathbf{FOL}}$-chains with the domain $\Sigma$.

PROOF. This is a generalisation of the proof of Lemma 20.

Notice that $card(\mathbb{S}en(\Sigma[C])) = \lambda$ and let $\{\rho_i \in \mathbb{S}en(\Sigma[C]) \mid 0 < i < \lambda\}$ be an enumeration of $\mathbb{S}en(\Sigma[C])$. We construct a $(\lambda, \mathcal{D}^{\mathbf{FOL}})$-chain using the enumeration $\{\rho_i \in \mathbb{S}en(\Sigma[C]) \mid 0 < i < \lambda\}$. We define

(1) $\Sigma_0 = \Sigma$, and for any successor ordinal $i < \lambda$ let $C^i$ be the finite set of all constants from $C$ that occur in $\rho_i$ but not in $\Sigma_{i-1}$, $\mathcal{L}_i = \Sigma_{i-1}[C^i]$, and $u_i = (\Sigma_{i-1} \hookrightarrow \Sigma_{i-1}[C^i])$.

   (a) If $\rho_i$ is an existentially quantified sentence, i.e. $\rho_i = (\exists X^i)\delta_i$, then by Lemma 20, $\Sigma \hookrightarrow \mathcal{L}_i[X^i]$ is a $\mathcal{D}^{\mathbf{FOL}}$-chain and we have $\Sigma \hookrightarrow \mathcal{L}_i[X^i] \in \mathcal{D}^{\mathbf{FOL}}_\Sigma$; let $K^i$ be a finite set of constants in $C$ that are different from the symbols in $\mathcal{L}_i$ such that there exists a bijection $\psi_i : X^i \to K^i$.

   (b) If $\rho_i$ is not an existentially quantified sentence then let $K^i = \emptyset$.

   We define $\Sigma_i = \mathcal{L}_i[K^i]$, $w_i = (\mathcal{L}_i \hookrightarrow \mathcal{L}_i[K^i])$ and $v_{i-1,i} = u_i; w_i$.

(2) for all limit ordinals $\tau \leq \lambda$, let $C^\tau = \bigcup_{0<i<\tau}(C^i \cup K^i)$, $\Sigma_\tau = \Sigma[C^\tau]$, and for all ordinals $i < \tau$ we define $v_{i,\tau} = (\Sigma_i \hookrightarrow \Sigma[C^\tau])$.

For all limit ordinals $\tau \leq \lambda$, since $C^\tau = \bigcup_{0<i<\tau}(C^i \cup K^i)$ and the sets $C^i$ and $K^i$ are finite for all successor ordinals $i < \tau$, we have that $card(C^\tau_s) \leq \tau$ for all sorts $s \in S$. Hence, for all limit ordinals $\tau < \lambda$ and sorts $s \in S$ we have $card(C^\tau_s) < card(C_s)$ which makes it possible to choose $K^i$ at $2(a)$ above. □

**Definition 23** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\Sigma \in |\mathbb{S}ig|$ a signature of power $\lambda$. A signature morphism $v : \Sigma \to \Sigma_\lambda$ is a *$\mathcal{D}$-extension* of $\Sigma$ via a $\Sigma$-model $M$ whenever $v$ is a $\mathcal{D}_\Sigma$-extension of $\Sigma$, where $\mathcal{D}_\Sigma$ consists of all $\mathcal{D}$-chains $v' : \Sigma \to \Sigma_{\lambda'}$ satisfying the following property: for every successor ordinal $i \leq \lambda'$, where $\lambda'$ is the length of the chain $v'$, and each $v'_{0,i-1}$-expansion $M_{i-1}$ of $M$ there exists a $v'_{i-1,i}$-expansion $M_i$ of $M_{i-1}$.

In our concrete examples of logical systems, $\mathcal{D}_\Sigma$ consists of extensions of $\Sigma$ with constants of sorts inhabited by the model $M$. [3]

[3] Given a first-order signature $(S, F, P)$ and a $(S, F, P)$-model $M$, the sort $s \in S$ is inhabited by the model $M$ if the carrier set $M_s$ is different from $\emptyset$.

**Lemma 24** Let $\Sigma = (S, F, P)$ be a **FOL**-signature of power $\lambda$, and $M$ a $\Sigma$-model. Consider a $S$-sorted set $C$ different from the symbols of $\Sigma$ such that $card(C_s) = \lambda$ for all sorts $s \in S$ inhabited by $M$ and $C_s = \emptyset$ for all sorts $s \in S$ which are not inhabited by $M$. Then $\nu : \Sigma \hookrightarrow \Sigma[C]$ is a $\mathcal{D}^{\mathbf{FOL}}$-extension of $\Sigma$ via $M$.

PROOF. Notice that $\mathcal{D}_\Sigma$ consists of signature extensions with sets of constants of sorts inhabited by $M$. Then the proof is similar to the proof of Lemma 22. □

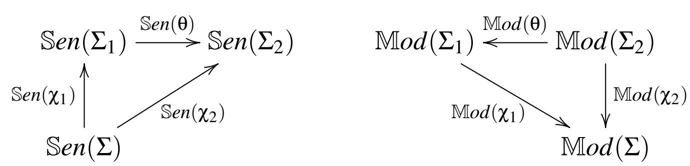
## 2.7 Substitutions

We recall the notion of substitution in institutions.

**Definition 25** [11] Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution and $\Sigma \in |\mathbb{S}ig|$. For any signature morphisms $\chi_1 : \Sigma \to \Sigma_1$ and $\chi_2 : \Sigma \to \Sigma_2$, a $\Sigma$-*substitution* $\theta : \chi_1 \to \chi_2$ consists of a pair $(\mathbb{S}en(\theta), \mathbb{M}od(\theta))$ , where

- $\mathbb{S}en(\theta) : \mathbb{S}en(\Sigma_1) \to \mathbb{S}en(\Sigma_2)$ is a function and
- $\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma_2) \to \mathbb{M}od(\Sigma_1)$ is a functor.

such that both of them preserve $\Sigma$, i.e. the following diagrams commute:




and such that the following *satisfaction condition* holds:

$$\mathbb{M}od(\theta)(M_2) \models \rho_1 \text{ iff } M_2 \models \mathbb{S}en(\rho_1)$$

for each $\Sigma_2$-model $M_2$ and each $\Sigma_1$-sentence $\rho_1$.

Note that a substitution $\theta : \chi_1 \to \chi_2$ is uniquely identified by its domain $\chi_1$, codomain $\chi_2$ and the pair $(\mathbb{S}en(\theta), \mathbb{M}od(\theta))$. We sometimes let $\_\restriction_\theta$ denote the functor $\mathbb{M}od(\theta)$, and let $\theta$ denote the sentence translation $\mathbb{S}en(\theta)$.

**Example 26 (FOL substitutions [11])** Consider two signature extensions with constants $\chi_1 : \Sigma \hookrightarrow \Sigma[C_1]$ and $\chi_2 : \Sigma \hookrightarrow \Sigma[C_2]$, where $\Sigma = (S, F, P) \in |\mathbb{S}ig^{\mathbf{FOL}}|$, $C_i$ is a set of constant symbols different from the the symbols in $\Sigma$. A function $\theta : C_1 \to T_\Sigma(C_2)$ represents a substitution between $\chi_1$ and $\chi_2$. On the syntactic side, $\theta$ can be canonically extended to a function $\mathbb{S}en(\theta) : \mathbb{S}en(\Sigma[C_1]) \to \mathbb{S}en(\Sigma[C_2])$ as follows:

- $\mathbb{S}en(\theta)(t_1 = t_2)$ is defined as $\theta^{term}(t) = \theta^{term}(t')$ for each $\Sigma[C_1]$-equation $t_1 = t_2$, where $\theta^{term} : T_\Sigma(C_1) \to T_\Sigma(C_2)$ is the unique extension of $\theta$ to a $\Sigma$-morphism.
- $\mathbb{S}en(\theta)(\pi(t_1, \ldots, t_n))$ is defined as $\pi(\theta^{term}(t_1), \ldots, \theta^{term}(t_n))$ for each $\Sigma[C_1]$-relational atom $\pi(t_1, \ldots, t_n)$.
- $\mathbb{S}en(\theta)(\vee E)$ is defined as $\vee \mathbb{S}en(\theta)(E)$ for each disjunction $\vee E$ of $\Sigma[C_1]$-sentences, and similarly for the case of any other Boolean connectives.

- $\mathbb{S}en(\theta)((\exists X)\rho)$ is defined as $(\exists X^{\theta})\mathbb{S}en(\theta')(\rho)$ for each $\Sigma[C_1]$-sentence $(\exists X)\rho$, where $X^{\theta} = \{(x,s,\Sigma[C_2]) \mid (x,s,\Sigma[C_1]) \in X\}$ and the substitution $\theta' : C_1 \cup X \to T_{\Sigma}(C_2 \cup X^{\theta})$ extends $\theta$ by mapping each variable $(x,s,\Sigma[C_1]) \in X$ to $(x,s,\Sigma[C_2]) \in X^{\theta}$.

On the semantics side, $\theta$ determines a functor $\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma[C_2]) \to \mathbb{M}od(\Sigma[C_1])$ such that for all $\Sigma[C_2]$-models $M$ we have

- $\mathbb{M}od(\theta)(M)_x = M_x$, for each sort $x \in S$, or operation symbol $x \in F$, or relation symbol $x \in P$, and
- $\mathbb{M}od(\theta)(M)_x = M_{\theta(x)}$ for each $x \in C_1$.

**Example 27 (PA-substitutions [7])** Consider two **PA** signature extensions with total constant symbols $\chi_1 : (S,TF,PF) \hookrightarrow (S,TF \cup C_1,PF)$ and $\chi_2 : (S,TF,PF) \hookrightarrow (S,TF \cup C_2,PF)$. Let $T_{(S,TF)}(C_i) \in \mathbb{M}od^{\mathbf{PA}}(S,TF,PF)$ be the partial algebra of terms formed with total constant symbols and elements from $C_i$. A $S$-sorted function $\theta : C_1 \to T_{(S,TF)}(C_2)$ represents a substitution between $\chi_1$ and $\chi_2$. On the syntactic side, $\theta$ can be canonically extended to a function $\mathbb{S}en(\theta) : \mathbb{S}en(S,TF \cup C_1,PF) \to \mathbb{S}en(S,TF \cup C_2,PF)$. On the semantics side, $\theta$ determines a functor $\mathbb{M}od(\theta) : \mathbb{M}od(S,TF \cup C_2,PF) \to \mathbb{M}od(S,TF \cup C_1,PF)$ such that for all $(S,TF \cup C_2,PF)$-models $M$ we have

- $\mathbb{M}od(\theta)(M)_x = M_x$, for each sort $x \in S$, or total operation symbol $x \in TF$, or partial operation symbol $x \in PF$, and
- $\mathbb{M}od(\theta)(M)_x = M_{\theta(x)}$ for each $x \in C_1$.

**Example 28 (HNK substitutions [6])** Consider two **HNK** signature extensions with operation symbols (of any type) $\chi_1 : (S,F) \hookrightarrow (S,F \cup C_1)$ and $\chi_2 : (S,F) \hookrightarrow (S,F \cup C_2)$. Let $T_{(S,F)}(C_i)$ be the $\overrightarrow{S}$-sorted set of terms with elements from $C_i$. A $\overrightarrow{S}$-sorted function $\theta : C_1 \to T_{(S,F)}(C_2)$ represents a substitution between $\chi_1$ and $\chi_2$. On the syntactic side, $\theta$ can be canonically extended to a function $\mathbb{S}en(\theta) : \mathbb{S}en(S,F \cup C_1) \to \mathbb{S}en(S,F \cup C_2)$. On the semantics side, $\theta$ determines a functor $\mathbb{M}od(\theta) : \mathbb{M}od(S,F \cup C_2) \to \mathbb{M}od(S,F \cup C_1)$ such that for all $(S,F \cup C_2)$-models $M$ we have

- $\mathbb{M}od(\theta)(M)_x = M_x$, for each sort $x \in S$, or function symbol $x \in F$, and
- $\mathbb{M}od(\theta)(M)_x = M_{\theta(x)}$ for each $x \in C_1$.

**Category of substitutions.** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution and $\Sigma \in |\mathbb{S}ig|$ a signature. $\Sigma$-substitutions form a category $\mathbb{S}ub^I(\Sigma)$, where the objects are signature morphisms $\Sigma \xrightarrow{\chi} \Sigma' \in |\Sigma/\mathbb{S}ig|$, and the arrows are substitutions $\theta : \chi_1 \to \chi_2$ as described in Definition 25. For any substitutions $\theta : \chi_1 \to \chi_2$ and $\theta' : \chi_2 \to \chi_3$ the composition $\theta;\theta'$ consists of the pair $(\mathbb{S}en(\theta;\theta'), \mathbb{M}od(\theta;\theta'))$, where $\mathbb{S}en(\theta;\theta') = \mathbb{S}en(\theta);\mathbb{S}en(\theta')$ and $\mathbb{M}od(\theta;\theta') = \mathbb{M}od(\theta');\mathbb{M}od(\theta)$.

Given a signature morphism $\varphi : \Sigma_0 \to \Sigma$ there exists a reduct functor $\mathbb{S}ub^I(\varphi) : \mathbb{S}ub^I(\Sigma) \to \mathbb{S}ub^I(\Sigma_0)$ that maps any $\Sigma$-substitution $\theta : \chi_1 \to \chi_2$ to the $\Sigma_0$-substitution $\mathbb{S}ub(\varphi)(\theta) : \varphi;\chi_1 \to \varphi;\chi_2$ such that $\mathbb{S}en(\mathbb{S}ub^I(\varphi)(\theta)) = \mathbb{S}en(\theta)$ and $\mathbb{M}od(\mathbb{S}ub^I(\varphi)(\theta)) = \mathbb{M}od(\theta)$. It follows that $\mathbb{S}ub^I : \mathbb{S}ig^{op} \to \mathbb{CAT}$ is a functor. In applications not all substitutions are of interest, and it is often assumed there is a substitution sub-functor $\mathbb{S}^I : \mathcal{D}^{op} \to \mathbb{CAT}$ of $\mathbb{S}ub^I : \mathbb{S}ig^{op} \to \mathbb{CAT}$ to work with, where $\mathcal{D} \subseteq \mathbb{S}ig$ is a subcategory of signature morphisms. When there is no danger of confusion we may drop the superscript $I$ from the notations.

**Assumption 29** Throughout this paper we assume that $\mathbb{S} : \mathcal{D}^{op} \to \mathbb{CAT}$ range over substitution functors with the following commutativity property of substitutions with the first-order constructors for sentences: for every signature $\Sigma \in |\mathbb{S}ig|$, each substitution $\theta : (\Sigma \stackrel{\chi_1}{\to} \Sigma_1) \to (\Sigma \stackrel{\chi_2}{\to} \Sigma_2) \in \mathbb{S}(\Sigma)$,

(1) $\theta(\neg e) = \neg\theta(e)$ for all $\Sigma_1$-sentences $\neg e$;

(2) $\theta(\vee E) = \vee\theta(E)$ for all $\Sigma_1$-sentences $\vee E$;

(3) for any $\Sigma_1$-sentence $(\exists\chi_1')e'$, where $\Sigma_1 \stackrel{\chi_1'}{\to} \Sigma_1' \in \mathcal{D}$, there exists $\Sigma_2 \stackrel{\chi_2'}{\to} \Sigma_2' \in \mathcal{D}$ and a substitution $\theta' : \chi_1;\chi_1' \to \chi_2;\chi_2' \in \mathbb{S}(\Sigma)$ such that the following diagrams are commutative

$$\begin{array}{ccc} \mathbb{S}en(\Sigma_1') & \xrightarrow{\mathbb{S}en(\theta')} & \mathbb{S}en(\Sigma_2') \\ \uparrow {\scriptstyle \mathbb{S}en(\chi_1')} & & \uparrow {\scriptstyle \mathbb{S}en(\chi_2')} \\ \mathbb{S}en(\Sigma_1) & \xrightarrow{\mathbb{S}en(\theta)} & \mathbb{S}en(\Sigma_2) \end{array} \qquad \begin{array}{ccc} \mathbb{M}od(\Sigma_1') & \xleftarrow{\mathbb{M}od(\theta')} & \mathbb{M}od(\Sigma_2') \\ \downarrow {\scriptstyle \mathbb{M}od(\chi_1')} & & \downarrow {\scriptstyle \mathbb{M}od(\chi_2')} \\ \mathbb{M}od(\Sigma_1) & \xleftarrow{\mathbb{M}od(\theta)} & \mathbb{M}od(\Sigma_2) \end{array}$$

and $\theta((\exists\chi_1')e') = (\exists\chi_2')\theta'(e')$.

The first two conditions of the above assumption trivially hold in concrete examples. The third condition is less obvious. In **FOL**, given a signature $\Sigma$, a substitution $\theta : C_1 \to T_\Sigma(C_2)$ and a $\Sigma[C_1]$-sentence $(\exists X)\rho$, the following diagram is commutative,

$$\begin{array}{ccc} C_1 \cup X & \xrightarrow{\theta'} & T_{\Sigma[C_2 \cup X^\theta]} \\ \uparrow & & \uparrow \\ C_1 & \xrightarrow[\theta]{} & T_{\Sigma[C_2]} \end{array}$$

where $X^\theta = \{(x,s,\Sigma[C_2]) \mid (x,s,\Sigma[C_1]) \in X\}$ and $\theta' : C_1 \cup X \to T_\Sigma(C_2 \cup X^\theta)$ extends $\theta$ by mapping each variable $(x,s,\Sigma[C_1]) \in X$ to $(x,s,\Sigma[C_2]) \in X^\theta$. It follows that the following digram is commutative

$$\begin{array}{ccc} T_{\Sigma[C_1 \cup X]} & \xrightarrow{\theta'^{term}} & T_{\Sigma[C_2 \cup X^\theta]} \\ \uparrow & & \uparrow \\ T_{\Sigma[C_1]} & \xrightarrow[\theta^{term}]{} & T_{\Sigma[C_2]} \end{array}$$

which implies that the third condition of Assumption 29 holds for first-order logic. This argument can be replicated for all logical systems presented in this paper.

**Example 30 (FOL substitution functor)** Given a signature $\Sigma \in |\mathbb{S}ig^{\mathbf{FOL}}|$, only $\Sigma$-substitutions represented by functions $\theta : C_1 \to T_\Sigma(C_2)$ are relevant for the present study, where $C_1$ and $C_2$ are finite sets of new constants for $\Sigma$. Let $\mathbb{S}^{\mathbf{FOL}} : (\mathcal{D}^{\mathbf{FOL}})^{op} \to \mathbb{CAT}$ denote the substitution functor which maps each signature $\Sigma$ to the subcategory of $\Sigma$-substitutions represented by functions of the form $\theta : C_1 \to T_\Sigma(C_2)$ as above.

**Example 31 (PA substitution functor)** Assume that $\mathcal{D}^{\mathbf{PA}} \subseteq \mathbb{S}ig^{\mathbf{PA}}$ is the broad subcategory of signature extensions with a finite number of total constants. Let $\mathbb{S}^{\mathbf{PA}} : (\mathcal{D}^{\mathbf{PA}})^{op} \to \mathbb{CAT}$ denote the sub-functor of $\mathbb{S}ub^{\mathbf{PA}} : (\mathbb{S}ig^{\mathbf{PA}})^{op} \to \mathbb{CAT}$ which maps each signature $(S, TF, PF)$ to the subcategory of $(S, TF, PF)$-substitutions represented by functions of the form $\theta : C_1 \to T_{(S,TF)}(C_2)$, where $C_1$ and $C_2$ are finite sets of total constants.

**Example 32 (HNK substitution functor)** Assume that $\mathcal{D}^{\mathbf{HNK}} \subseteq \mathbb{S}ig^{\mathbf{HNK}}$ is the broad subcategory of signature extensions with a finite number of function symbols of any type. Let $\mathbb{S}^{\mathbf{HNK}} : (\mathcal{D}^{\mathbf{HNK}})^{op} \to \mathbb{CAT}$ denote the sub-functor of $\mathbb{S}ub^{\mathbf{HNK}} : (\mathbb{S}ig^{\mathbf{HNK}})^{op} \to \mathbb{CAT}$ which maps each signature $(S, F)$ to the subcategory of $(S, F)$-substitutions represented by functions $\theta : C_1 \to T_{(S,F)}(C_2)$, where $C_1$ and $C_2$ are finite sets of new function symbols for $(S, F)$ of any type.

**Proposition 33** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution. Assume two presentation morphisms $(\Sigma, E) \xrightarrow{\chi_1} (\Sigma_1, E_1)$ and $(\Sigma, E) \xrightarrow{\chi_2} (\Sigma_2, E_2)$ such that $E_1 \models\!| \chi_1(E)$ and $E_2 \models\!| \chi_2(E)$. If $\theta : (\Sigma \xrightarrow{\chi_1} \Sigma_1) \to (\Sigma \xrightarrow{\chi_2} \Sigma_2)$ is a substitution in $I$ then $\theta : ((\Sigma, E) \xrightarrow{\chi_1} (\Sigma_1, E_1)) \to ((\Sigma, E) \xrightarrow{\chi_2} (\Sigma_2, E_2))$ is a substitution in $I^{pres}$, where

1. $\mathbb{S}en(\Sigma_1, E_1) \xrightarrow{\mathbb{S}en(\theta)} \mathbb{S}en(\Sigma_2, E_2) = \mathbb{S}en(\Sigma_1) \xrightarrow{\mathbb{S}en(\theta)} \mathbb{S}en(\Sigma_2)$ and
2. $\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma_2, E_2) \to \mathbb{M}od(\Sigma_1, E_1)$ is obtained from the functor $\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma_2) \to \mathbb{M}od(\Sigma_1)$ by restriction to $\mathbb{M}od(\Sigma_2, E_2)$ and corestriction to $\mathbb{M}od(\Sigma_1, E_1)$.

PROOF. Since $\mathbb{S}en(\Sigma_i, E_i) = \mathbb{S}en(\Sigma_i)$, where $i \in \{1, 2\}$, the definition of $\mathbb{S}en(\theta) : \mathbb{S}en(\Sigma_1, E_1) \to \mathbb{S}en(\Sigma_2, E_2)$ is consistent. We prove that $\mathbb{M}od(\theta)(\mathbb{M}od(\Sigma_2, E_2)) \subseteq \mathbb{M}od(\Sigma_1, E_1)$ which implies that the definition of $\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma_2, E_2) \to \mathbb{M}od(\Sigma_1, E_1)$ is consistent too. Since $\chi_1(E) \models\!| E_1$, by the satisfaction condition for substitutions, $\theta(\chi_1(E)) = \chi_2(E) \models\!| \theta(E_1)$; since $\chi_2(E) \models\!| E_2$, we obtain $\theta(E_1) \models\!| E_2$; it follows that for all $M_2 \in |\mathbb{M}od(\Sigma_2, E_2)|$ we have $M_2\!\upharpoonright_\theta \in |\mathbb{M}od(\Sigma_1, E_1)|$, which implies $\mathbb{M}od(\theta)(\mathbb{M}od(\Sigma_2, E_2)) \subseteq \mathbb{M}od(\Sigma_1, E_1)$. Hence, the definition of $\theta$ in $I^{pres}$ is consistent. $\square$

Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution and $\mathcal{D} \subseteq \mathbb{S}ig$ a subcategory of signature morphisms. By Proposition 33, each substitution functor $\mathbb{S}^I : \mathcal{D}^{op} \to \mathbb{CAT}$ can be extended to a substitution functor $\mathbb{S}^{I^{pres}} : \mathcal{D}^{op} \to \mathbb{CAT}$ in $I^{pres}$:

- for each $(\Sigma, E) \in |\mathbb{S}ig^{I^{pres}}|$, $\mathbb{S}^{I^{pres}}(\Sigma, E)$ is the subcategory of substitutions $\theta : ((\Sigma, E) \xrightarrow{\chi_1} (\Sigma_1, E_1)) \to ((\Sigma, E) \xrightarrow{\chi_2} (\Sigma_2, E_2))$ such that $(\Sigma, E) \xrightarrow{\chi_i} (\Sigma_i, E_i) \in \mathcal{D}$ and $\theta : (\Sigma \xrightarrow{\chi_1} \Sigma_1) \to (\Sigma \xrightarrow{\chi_2} \Sigma_2) \in \mathbb{S}^I(\Sigma)$.
- for each $(\Sigma_0, E_0) \xrightarrow{\chi} (\Sigma, E) \in \mathcal{D}$, the functor $\mathbb{S}^{I^{pres}}(\chi) : \mathbb{S}^{I^{pres}}(\Sigma, E) \to \mathbb{S}^{I^{pres}}(\Sigma_0, E_0)$ is obtained from the functor $\mathbb{S}ub^{I^{pres}}(\chi) : \mathbb{S}ub^{I^{pres}}(\Sigma, E) \to \mathbb{S}ub^{I^{pres}}(\Sigma_0, E_0)$ by restriction to $\mathbb{S}^{I^{pres}}(\Sigma, E)$ and corestriction to $\mathbb{S}^{I^{pres}}(\Sigma_0, E_0)$.

For the sake of simplicity we may denote $\mathbb{S}^{I^{pres}}$ simply by $\mathbb{S}$.

## 2.8 Reachable Models

We give an institution-independent characterisation of the models that consist of interpretations of terms.

**Definition 34** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S} : \mathcal{D}^{op} \to \mathbb{CAT}$ a substitution functor. A model $M \in |\mathbb{M}od(\Sigma)|$, where $\Sigma \in |\mathbb{S}ig|$, is $\mathbb{S}$-*reachable* if for every signature morphism $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$ and each $\chi$-expansion $M'$ of $M$ there exists a substitution $\theta : \chi \to 1_\Sigma \in \mathbb{S}(\Sigma)$ such that for all morphisms $M' \xrightarrow{h'} N' \in \mathbb{M}od(\Sigma')$ we have $(h' \upharpoonright_\chi) \upharpoonright_\theta = h'$.

This notion of reachable model is a generalisation of the one in [24] to abstract substitutions. In this subsection we study the notion of reachability in concrete logical systems.

**Proposition 35** In **FOL**, a model is $\mathbb{S}^{\mathbf{FOL}}$-reachable iff its elements consist of interpretations of terms.

PROOF. Let $\Sigma = (S, F, P)$ be a first-order signature and $M \in |\mathbb{M}od^{\mathbf{FOL}}(\Sigma)|$ a model which consists of interpretation of terms, i.e. the unique morphism $T_\Sigma \xrightarrow{h_M} M$ given by the initiality of $T_\Sigma$ is surjective. Let $\chi : \Sigma \hookrightarrow \Sigma[C]$ be a signature extension with a finite number of constants, and $M'$ a $\chi$-expansion of $M$. The model $M'$ consists of a pair $(M, f)$, where $f : C \to M$ is defined by $f(c) = M'_c$ for all $c \in C$. Since $T_\Sigma \xrightarrow{h_M} M$ is surjective, there exists $\theta : C \to T_\Sigma$ such that $\theta; h_M = f$. Now, let $M' \xrightarrow{h'} N'$ be a $\Sigma[C]$-morphism. $N'$ consists of a pair $(N, g)$, where $N \in |\mathbb{M}od(\Sigma)|$ and $g : C \to N$ is defined by $g(c) = N'_c$ for all $c \in C$. Note that $h'$ consists of a $\Sigma$-morphism $M \xrightarrow{h} N$ such that $f; h = g$. Let $T_\Sigma \xrightarrow{h_N} N$ be the unique $\Sigma$-morphism given by the initiality of $T_\Sigma$. We have $(M \xrightarrow{h} N)\upharpoonright_\theta = ((M, \theta; h_M) \xrightarrow{h} (N, \theta; h_N))$. Since $h_N = h_M; h$, we have $\theta; h_N = \theta; h_M; h = f; h = g$. We obtain $(M \xrightarrow{h} N)\upharpoonright_\theta = ((M, \theta; h_M) \xrightarrow{h} (N, \theta; h_N)) = ((M, f) \xrightarrow{h} (N, g)) = (M' \xrightarrow{h'} N')$ . Hence, $(h' \upharpoonright_\chi)\upharpoonright_\theta = h \upharpoonright_\theta = h'$.

Assume that $M \in |\mathbb{M}od^{\mathbf{FOL}}(\Sigma)|$ is $\mathbb{S}^{\mathbf{FOL}}$-reachable, where $\Sigma \in |\mathbb{S}ig^{\mathbf{FOL}}|$, and let $m \in M$. We show that there exists $t \in T_\Sigma$ such that $M_t = m$. Consider a new constant $c$ for $\Sigma$, and let $M'$ be an expansion of $M$ along $\chi : \Sigma \hookrightarrow \Sigma[c]$ such that $M'_c = m$. Since $M$ is $\mathbb{S}^{\mathbf{FOL}}$-reachable, there exists a substitution $\theta : \{c\} \to T_\Sigma$ such that $M\upharpoonright_\theta = M'$. We have $M_{\theta(c)} = (M\upharpoonright_\theta)_c = M'_c = m$. □

The proof of Proposition 35 is a slight generalisation of the one in [24], and it is included in this paper for the convenience of the reader. One important consequence of the above proposition is the following corollary.

**Corollary 36** In **FOL**, for any set $B \subseteq \mathbb{S}en^{\mathbf{FOL}}(\Sigma)$ of atomic sentences, where $\Sigma \in |\mathbb{S}ig^{\mathbf{FOL}}|$, there exists a basic model $M_B$ of $B$ which is $\mathbb{S}^{\mathbf{FOL}}$-reachable.

Another consequence of the fact that reachable models consist of interpretations of terms is the corollary below.

**Corollary 37** For any $\mathbb{S}^{\mathbf{FOL}}$-reachable model $M \in |\mathbb{M}od^{\mathbf{FOL}}(\Sigma)|$, where $\Sigma \in |\mathbb{S}ig^{\mathbf{FOL}}|$, we have $card(M) \leq card(\mathbb{S}en^{\mathbf{FOL}}(\Sigma))$.

Similar results as Proposition 35, Corollary 36 and 37 hold for **POA** too.

**Proposition 38** In **PA**, a partial algebra is $\mathbb{S}^{\mathbf{PA}}$-reachable if its elements consist of interpretations of terms formed with total operation symbols.

PROOF. Let $(S,TF,PF) \in |\mathbb{S}ig^{\mathbf{PA}}|$ and $M \in |\mathbb{M}od^{\mathbf{PA}}(S,TF,PF)|$ such that the unique $(S,TF,PF)$-homomorphism $T_{(S,TF)} \stackrel{h_M}{\to} M$ is surjective. Let $\chi : (S,TF,PF) \hookrightarrow (S,TF \cup C,PF)$ be a signature extension with a finite number of total constants, and $M'$ a $\chi$-expansion of $M$. Note that $M'$ consists of a pair $(M,f)$, where $f : C \to M$ is defined by $f(c) = M'_c$ for all $c \in C$. Since the unique $(S,TF,PF)$-morphism $T_{(S,TF)} \stackrel{h_M}{\to} M$ is surjective, there exists $\theta : C \to T_{(S,TF)}$ such that $\theta;h_M = f$. Now, let $M' \stackrel{h'}{\to} N' \in \mathbb{M}od^{\mathbf{PA}}(S,TF \cup C,PF)$. $N'$ consists of a pair $(N,g)$, where $N \in |\mathbb{M}od^{\mathbf{PA}}(S,TF,PF)|$ and $g : C \to N$ is defined by $g(c) = N'_c$ for all $c \in C$. Note that $h'$ consists of a $(S,TF,PF)$-morphism $M \stackrel{h}{\to} N$ such that $f;h = g$. Let $T_{(S,TF)} \stackrel{h_N}{\to} N$ be the unique $(S,TF,PF)$-morphism given by the initiality of $T_{(S,TF)}$. We have $(M \stackrel{h}{\to} N)\upharpoonright_\theta = ((M,\theta;h_M) \stackrel{h}{\to} (N,\theta;h_N))$. Since $T_{(S,TF)}$ is the initial partial algebra, $h_N = h_M;h$ and we get $\theta;h_N = \theta;h_M;h = f;h = g$. It follows that $(M \stackrel{h}{\to} N)\upharpoonright_\theta = ((M,\theta;h_M) \stackrel{h}{\to} (N,\theta;h_N)) = ((M,f) \stackrel{h}{\to} (N,g)) = (M' \stackrel{h'}{\to} N')$ . Hence, $(h'\upharpoonright_\chi)\upharpoonright_\theta = h\upharpoonright_\theta = h'$.

For the converse implication, assume that $M \in |\mathbb{M}od^{\mathbf{PA}}(S,TF,PF)|$ is $\mathbb{S}^{\mathbf{PA}}$-reachable, and let $m \in M$. We show that there exists $t \in T_{(S,TF)}$ such that $M_t = m$. Consider a new constant $c$ for $(S,TF,PF)$, and let $M'$ be an expansion of $M$ along the inclusion $\chi : (S,TF,PF) \hookrightarrow (S,TF \cup \{c\},PF)$ such that $M'_c = m$. Since $M$ is $\mathbb{S}^{\mathbf{PA}}$-reachable, there exists a substitution $\theta : \{c\} \to T_{(S,TF)}$ such that $M\upharpoonright_\theta = M'$. We have that $M_{\theta(c)} = (M\upharpoonright_\theta)_c = M'_c = m$. $\square$

In **PA**, the basic models corresponding to the sets of atomic sentences consist of interpretations of terms formed with both total and partial operation symbols. It follows that they are not $\mathbb{S}^{\mathbf{PA}}$-reachable.

**Proposition 39** In **HNK**, a model $M$ is $\mathbb{S}^{\mathbf{HNK}}$-reachable if its elements consist of interpretations of terms.

PROOF. Assume that the elements of $M$ consist of interpretations of terms. Let $\chi : (S,F) \hookrightarrow (S,F \cup C) \in \mathcal{D}^{\mathbf{HNK}}$, where $C$ is a finite set of function symbols, and $M'$ be a $\chi$-expansion of $M$. Note that $M'$ consists of a pair $(M,f)$, where $M \in |\mathbb{M}od^{\mathbf{HNK}}(S,F)|$ and $f = (C_s \stackrel{f_s}{\to} M_s)_{s \in \overrightarrow{S}}$ is a $\overrightarrow{S}$-sorted function defined by $f(c) = M'_c$ for all $c \in C$. Each term in $T_{(S,F)}$ has its own unique interpretation in $M$. There exists a surjective function $f_M = ((T_{(S,F)})_s \stackrel{(f_M)_s}{\to} M_s)_{s \in \overrightarrow{S}}$. It follows that there exists a substitution $\theta : C \to T_{(S,F)}$ such that $\theta;f_M = f$. Let $M' \stackrel{h'}{\to} N' \in \mathbb{M}od^{\mathbf{HNK}}(S,F \cup C)$. Assume that $N' = (N,g)$, where $N \in |\mathbb{M}od^{\mathbf{HNK}}(S,F)|$ and $g = (C_s \stackrel{g_s}{\to} N_s)_{s \in \overrightarrow{S}}$ is defined by $g(c) = N'_c$ for all $c \in C$. Note that $h'$ consists of a $(S,F)$-morphism $M \stackrel{h}{\to} N$ such that $f;h = g$. Let $f_N = ((T_{(S,F)})_s \stackrel{(f_N)_s}{\to} N_s)_{s \in \overrightarrow{S}}$ be the mapping given by the unique interpretation of terms into $N$. It follows that $f_N = f_M;h$. We have $(M \stackrel{h}{\to} N)\upharpoonright_\theta = ((M,\theta;f_M) \stackrel{h}{\to} (N,\theta;f_N)$. Since $\theta;f_N = \theta;f_M;h = f;h = g$, we obtain $(M \stackrel{h}{\to} N)\upharpoonright_\theta = ((M,f) \stackrel{h}{\to} (N,g))$. Hence, $(h'\upharpoonright_\chi)\upharpoonright_\theta = h\upharpoonright_\theta = h'$.

For the converse implication, assume that $M \in |\mathbb{M}od^{\mathbf{HNK}}(S,F)|$ is $\mathbb{S}^{\mathbf{HNK}}$-reachable, and let $m \in M$. We show that there exists $t \in T_{(S,F)}$ such that $M_t = m$. Consider a constant $c$ different from the symbols in $F$, and let $M'$ be the expansion of $M$ along $\chi : (S,F) \hookrightarrow (S,F \cup \{c\})$ such that $M'_c = m$. Since $M$ is $\mathbb{S}^{\mathbf{HNK}}$-reachable there exists a substitution $\theta : \{c\} \to T_{(S,F)}$ such that $M\upharpoonright_\theta = M'$. We have that $M_{\theta(c)} = (M\upharpoonright_\theta)_c = M'_c = m$. $\square$

One can easily show that if a model is reachable in a given institution then it is is reachable in the corresponding institution of presentations.

**Proposition 40** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S} : \mathcal{D}^{op} \to \mathbb{CAT}$ a substitution functor. For any presentation $(\Sigma, E) \in |\mathbb{S}ig^{I^{pres}}|$, a model $M \in |\mathbb{M}od(\Sigma, E)|$ is $\mathbb{S}^{I^{pres}}$-reachable if it is $\mathbb{S}^{I}$-reachable.

PROOF. Assume that $M \in |\mathbb{M}od(\Sigma, E)|$ is $\mathbb{S}^I$-reachable. Consider a presentation morphism $(\Sigma, E) \xrightarrow{\chi} (\Sigma', E') \in \mathcal{D}$ and let $M'$ be a $\chi$-expansion of $M$. Since $M$ is $\mathbb{S}^I$-reachable, there exists a substitution $\theta : (\Sigma \xrightarrow{\chi} \Sigma') \to (\Sigma \xrightarrow{1_\Sigma} \Sigma) \in \mathbb{S}(\Sigma)$ such that for all $M' \xrightarrow{h'} N' \in \mathbb{M}od(\Sigma')$ we have $(h'\restriction_\chi)\restriction_\theta = h'$. It follows that $\theta : ((\Sigma, E) \xrightarrow{\chi} (\Sigma', E')) \to ((\Sigma, E) \xrightarrow{1_\Sigma} (\Sigma, E)) \in \mathbb{S}^{I^{pres}}(\Sigma, E)$. Since $(h'\restriction_\chi)\restriction_\theta = h'$ for all $M' \xrightarrow{h'} N' \in \mathbb{M}od(\Sigma')$, we have $(h'\restriction_\chi)\restriction_\theta = h'$ for all $M' \xrightarrow{h'} N' \in \mathbb{M}od(\Sigma', E')$.
□

## 2.9 Elementary Morphisms

In classical model theory [4], an injective model morphism $h : M \to N$ is called an *elementary embedding* if one of the following equivalent conditions holds:

- for each formula $\rho(x_1, \ldots, x_n)$ and each sequence $m_1, \ldots, m_n \in M$, $M \models \rho(m_1, \ldots, m_n)$ iff $N \models \rho(h(m_1), \ldots, h(m_n))$.
- for each formula $\rho(x_1, \ldots, x_n)$ and each sequence $m_1, \ldots, m_n \in M$, $M \models \rho(m_1, \ldots, m_n)$ implies $N \models \rho(h(m_1), \ldots, h(m_n))$.

The institutional generalisation of this concept interprets elementary embedding in the following way: the morphism $h$ preserves satisfaction of sentences in any language extending with constants the original language, regardless of the interpretation of these constants.

**Definition 41** [26] Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, and $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms. A $\Sigma$-morphism $M \xrightarrow{h} N$ is called *$\mathcal{D}$-elementary*, where $\Sigma$ is a signature, if all expansions of $h$ along signature morphisms in $\mathcal{D}$ preserve satisfaction of sentences (i.e. for every signature morphism $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$, each $\chi$-expansion $M' \xrightarrow{h'} N'$ of $h$, and any sentence $\rho \in \mathbb{S}en(\Sigma')$, it holds that $M' \models_{\Sigma'} \rho$ implies $N' \models_{\Sigma'} \rho$).

In our concrete examples of institutions $\mathcal{D}$ consists of signature extensions with a finite number of constants. In **FOL**, given an elementary embedding $h : M \to N$, note that $M$ is isomorphic to $h(M)$, and $h(M)$ is called an *elementary submodel* of $N$.

**Proposition 42** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be a semi-exact institution that has pushouts of signature morphisms, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms closed under pushouts, and $\mathbb{S} : \mathcal{D}^{op} \to \mathbb{CAT}$ a substitution functor. Consider a signature morphism $\Sigma \xrightarrow{\nu} \Sigma_1 \in \mathbb{S}ig$, a $\Sigma$-morphism $h : M \to N$, and a $\nu$-expansion $h_1 : M_1 \to N_1$ of $h$.

1. Assuming that $M_1$ is $\mathbb{S}$-reachable, if $h_1$ preserves satisfaction of sentences then $h$ is $\mathcal{D}$-elementary.

2. Assuming that for any $\rho \in \mathbb{S}en(\Sigma_1)$ there exists $\Sigma \xrightarrow{d_\rho} \Sigma_\rho \in \mathcal{D}$, $\Sigma_\rho \xrightarrow{v_\rho} \Sigma_1 \in \mathbb{S}ig$ and $e \in \mathbb{S}en(\Sigma_\rho)$ such that $v = d_\rho; v_\rho$ and $\rho = v_\rho(e)$, if $h$ is $\mathcal{D}$-elementary then $h_1$ preserves satisfaction of sentences.

PROOF.

1. We assume that $M_1$ is $\mathbb{S}$-reachable and $h_1$ preserves the satisfaction of sentences. Let $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$ and $M' \xrightarrow{h'} N'$ a $\chi$-expansion of $h$. We show that $h'$ preserves satisfaction of sentences.

   Let $\delta \in \mathbb{S}en(\Sigma')$ such that $M' \models_{\Sigma'} \delta$. Consider the following pushout

$$\begin{array}{ccc} \Sigma' & \overset{v'}{\dashrightarrow} & \Sigma'_1 \\ \chi \uparrow & & \uparrow \chi_1 \\ \Sigma & \underset{v}{\longrightarrow} & \Sigma_1 \end{array}$$

   such that $\chi_1 \in \mathcal{D}$. By the semi-exactness of $I$, there exists a $\Sigma'_1$-homomorphism $h'_1 : M'_1 \to N'_1$ such that $h'_1\upharpoonright_{v'} = h'$ and $h'_1\upharpoonright_{\chi_1} = h_1$. By the satisfaction condition, $M'_1 \models_{\Sigma'_1} v'(\delta)$. Since $M_1$ is $\mathbb{S}$-reachable and $\chi_1 \in \mathcal{D}$ there exists a substitution $\theta : \chi_1 \to 1_{\Sigma_1}$ such that $(h'_1\upharpoonright_{\chi_1})\upharpoonright_\theta = h'_1$. By the satisfaction condition for substitutions, $M_1 \models_{\Sigma_1} \theta(v'(\delta))$. Since $M_1 \xrightarrow{h_1} N_1$ preserves satisfaction of sentences, $N_1 \models_{\Sigma_1} \theta(v'(\delta))$. By the satisfaction condition for substitutions, $N'_1 \models_{\Sigma'_1} v'(\delta)$. By the satisfaction condition for the institution $I$, we obtain $N' \models_{\Sigma'} \delta$.

2. We assume that $h$ is $\mathcal{D}$-elementary and for any $\rho \in \mathbb{S}en(\Sigma_1)$ there exists $\Sigma \xrightarrow{d_\rho} \Sigma_\rho \in \mathcal{D}$, $\Sigma_\rho \xrightarrow{v_\rho} \Sigma_1 \in \mathbb{S}ig$ and $e \in \mathbb{S}en(\Sigma_\rho)$ such that $v = d_\rho; v_\rho$ and $\rho = v_\rho(e)$. Let $\rho \in \mathbb{S}en(\Sigma_1)$ such that $M_1 \models_{\Sigma_1} \rho$. We prove that $N_1 \models_{\Sigma_1} \rho$.

   We have $(\Sigma \xrightarrow{v} \Sigma_1) = (\Sigma \xrightarrow{d_\rho} \Sigma_\rho); (\Sigma_\rho \xrightarrow{v_\rho} \Sigma_1)$ and $\rho = v_\rho(e)$ for some $d_\rho \in \mathcal{D}$, $v_\rho \in \mathbb{S}ig$ and $e \in \mathbb{S}en(\Sigma_\rho)$. By the satisfaction condition, $M_1\upharpoonright_{v_\rho} \models_{\Sigma_\rho} e$. Since $h$ is $\mathcal{D}$-elementary and $h = (h_1\upharpoonright_{v_\rho})\upharpoonright_{d_\rho}$, we have $N_1\upharpoonright_{v_\rho} \models_{\Sigma_\rho} e$. By the satisfaction condition, we obtain $N_1 \models_{\Sigma_1} \rho$.

□

Consider a first-order signature extension with constants $\Sigma \hookrightarrow \Sigma[C_1]$, and assume that $\rho \in \mathbb{S}en^{\mathbf{FOL}}(\Sigma[C_1])$. Let $C_\rho$ be the finite set of all constants occurring in $\rho$. Note that $\rho \in \mathbb{S}en(\Sigma[C_\rho])$, $\Sigma \hookrightarrow \Sigma[C_1] = (\Sigma \hookrightarrow \Sigma[C_\rho]); (\Sigma[C_\rho] \hookrightarrow \Sigma[C_1])$, and $\Sigma \hookrightarrow \Sigma[C_\rho] \in \mathcal{D}^{\mathbf{FOL}}$. By Proposition 35 and 42, in **FOL**, a $\Sigma$-morphism $M \xrightarrow{h} N$ is $\mathcal{D}^{\mathbf{FOL}}$-elementary iff the $\Sigma_M$-morphism $M_M \xrightarrow{h_M} N_M$ preserves satisfaction of sentences, where $\Sigma_M$ is the extension of $\Sigma$ with the elements of $M$, $M_M$ is the expansion of $M$ along $\Sigma \hookrightarrow \Sigma_M$ interpreting each $m \in M$ as $m$, $N_M$ is the expansion of $N$ along $\Sigma \hookrightarrow \Sigma_M$ interpreting each $m \in M$ as $h(m)$, and $h_M$ is the obvious expansion of $h$ along $\Sigma \hookrightarrow \Sigma_M$.

# 3 Downward Löwenheim-Skolem Theorem

In this section we formalise the DLSP in institution theory and we prove the DLST in an arbitrary institution satisfying certain logic properties.

In **FOL**, the *Löwenheim number* of a signature $(S,F,P)$ is the least cardinal $\lambda$ such that any model has an elementary submodel of power at most $\lambda$. Our abstract infrastructure allows to define an upper bound for the cardinality of a model using the notion of reachability. Hence, one can define the Löwenheim number for any institution.

**Definition 43 (Löwenheim number)** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S} : \mathcal{D}^{op} \to \mathbb{CAT}$ a substitution functor. The $\mathbb{S}$-*Löwenheim number* of a signature $\Sigma \in |\mathbb{S}ig|$, is the least cardinal $\lambda$ such that for any $\Sigma$-model $M$ there exists

- a $\mathcal{D}$-chain $v : \Sigma \to \Sigma'$ such that the power of $\Sigma'$ is $\lambda$, and
- a $\Sigma'$-morphism $h' : N' \to M'$ such that $N'$ is $\mathbb{S}$-reachable, $M'$ is a $v$-expansion of $M$, and $h'$ preserves the satisfaction of sentences.

According to Proposition 42, the model morphism $h = h'\restriction_v$ in Definition 43 is $\mathcal{D}$-elementary. If we instantiate Definition 43 to **FOL** then since $N'$ is reachable, $card(N') \leq card(\mathbb{S}en(\Sigma')) = \lambda$, which implies $card(N') = card(N'\restriction_v) \leq \lambda$; it follows that $M_\lambda = h(N'\restriction_v)$ is an elementary submodel of $M$ and $card(M_\lambda) \leq \lambda$.

**Definition 44 (DLSP)** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S} : \mathcal{D}^{op} \to \mathbb{CAT}$ a substitution functor. $I$ has the $\mathbb{S}$-*DLSP* if the $\mathbb{S}$-Löwenheim number of any signature is equal to its power.

However, Classical DLST says that given a first-order signature $(S,F,P)$, for each $(S,F,P)$-model $M$ and any cardinal $\lambda$ such that $card(\mathbb{S}en(S,F,P)) \leq \lambda < card(M)$, there exists an elementary submodel $M_\lambda$ of $M$ such that $card(M_\lambda) = \lambda$. We show that this result is a direct consequence of DLSP.

**Lemma 45** If **FOL** has the $\mathbb{S}^{\mathbf{FOL}}$-DLSP then for every **FOL** signature $\Sigma = (S,F,P)$, all $\Sigma$-models $M$, and any cardinal $\lambda$ such that $card(\mathbb{S}en(\Sigma)) \leq \lambda < card(M)$ there exists an elementary submodel $M_\lambda$ of $M$ of cardinality $\lambda$.

PROOF. Assume a first-order signature $\Sigma = (S,F,P)$, a $\Sigma$-model $M$, and a cardinal $\lambda$ such that $card(\mathbb{S}en(\Sigma)) \leq \lambda < card(M)$. Consider a $S$-sorted set $C$ such that $card(C) = \lambda$ and $C_s \subseteq M_s$ for all sorts $s \in S$. Let $M_C$ be the expansion of $M$ along $u : \Sigma \hookrightarrow \Sigma[C]$ interpreting each $c \in C$ as $c$. Notice that $card(\mathbb{S}en(\Sigma[C])) = \lambda$. Since we assumed that **FOL** has DLSP, there exists an elementary embedding $h_C : N \to M_C$ such that $card(N) \leq \lambda$. Since $h_C$ is elementary and $M_C \models \neg(c_1 = c_2)$ for all $c_1, c_2 \in C$ such that $c_1 \neq c_2$, we obtain $N \models \neg(c_1 = c_2)$ for all $c_1, c_2 \in C$ such that $c_1 \neq c_2$. It follows that $\lambda = card(C) \leq card(N)$, which implies $card(N) = \lambda$. Finally, note that $h = h_C\restriction_u$ is also an elementary embedding, $M_\lambda = h(N\restriction_u)$ is an elementary submodel of $M$ and $card(M_\lambda) = card(N\restriction_u) = card(N) = \lambda$. □

The argument used in the proof of the above lemma can be replicated for all examples of institutions given in this paper. We focus on proving an abstract version of DLST.

**Theorem 46 (DLST)** Consider an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, a broad subcategory $\mathcal{Q} \subseteq \mathbb{S}ig$ of signature morphisms, and a sub-functor $\mathbb{S}en^0 \subseteq \mathbb{S}en$ such that

1. all sentences of $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ are constructed from the sentences of $I^0 = (\mathbb{S}ig, \mathbb{S}en^0, \mathbb{M}od, \models)$ by applying Boolean connectives and quantification over the signature morphisms in $\mathcal{Q}$,

2. for all $\Sigma \in |\mathbb{S}ig|$ and $E^0 \subseteq \mathbb{S}en^0(\Sigma)$ we have $\mathbb{S}en^0(\Sigma) \cap \mathtt{cl}(E^0) = E^0$, where $\mathtt{cl}(E^0)$ is the least set of $\Sigma$-sentences obtained from $E^0$ by applying Boolean connectives and quantification over the signature morphisms in $\mathcal{Q}$, [4] and

3. $I$ is inductive-exact.

Let $\mathcal{D}$ be a broad subcategory of signature morphisms such that $\mathcal{Q} \subseteq \mathcal{D}$ and

4. for all $\Sigma \in |\mathbb{S}ig|$ and $M \in |\mathbb{M}od(\Sigma)|$, there exists a $\mathcal{D}$-extension of $\Sigma$ via $M$.

Let $\mathbb{S} : \mathcal{D}^{op} \to \mathbb{CAT}$ be a substitution functor such that

5. for all signatures $\Sigma \in |\mathbb{S}ig|$ and substitutions $\theta : (\Sigma \xrightarrow{\chi_1} \Sigma_1) \to (\Sigma \xrightarrow{\chi_2} \Sigma_2) \in \mathbb{S}(\Sigma)$, we have $\theta(\mathbb{S}en^0(\Sigma_1)) \subseteq \mathbb{S}en^0(\Sigma_2)$, and

6. each set of sentences in $I^0$ is basic and it has a basic $\mathbb{S}$-reachable model.

Then $I$ has $\mathbb{S}$-DLSP. Moreover, if $v : \Sigma \to \Sigma_\lambda$ is a $\mathcal{D}$-extension of $\Sigma$ via $M \in |\mathbb{M}od(\Sigma)|$ then there exists a $\Sigma_\lambda$-morphism $h_\lambda : N_\lambda \to M_\lambda$ such that $N_\lambda$ is $\mathbb{S}$-reachable, $M_\lambda\upharpoonright_v = M$, and $h_\lambda$ preserves satisfaction of sentences.

PROOF. Let $v : \Sigma \to \Sigma_\lambda$ be a $\mathcal{D}$-extension of $\Sigma$ via $M \in |\mathbb{M}od(\Sigma)|$. Let $\{\rho_i \in \mathbb{S}en(\Sigma_\lambda) \mid 0 < i < \lambda\}$ be the enumeration of $\mathbb{S}en(\Sigma_\lambda)$ such that all conditions of Definition 23 are met for $v$. We construct a chain of pairs $((\Sigma_i, \Gamma_i), M_i)_{i \leq \lambda}$, where $(\Sigma_i, \Gamma_i) \in |\mathbb{S}ig^{I^{pres}}|$ and $M_i \in |\mathbb{M}od(\Sigma_i, \Gamma_i)|$, such that

(1) for all successor ordinals $i > 0$, $v_{i-1,i}(\Gamma_{i-1}) \subseteq \Gamma_i$ and $M_i\upharpoonright_{v_{i-1,i}} = M_{i-1}$,

(2) for all limit ordinals $\tau \leq \lambda$, $\Gamma_\tau = \cup_{i<\tau} v_{i,\tau}(\Gamma_i)$ and $M_\tau\upharpoonright_{v_{i,\tau}} = M_i$ for all $i < \tau$, and

(3) $\Gamma_\lambda$ is maximal consistent.

We proceed as follows:

(1) We define $\Gamma_0 = \emptyset$ and $M_0 = M$. Assume that we have defined the pair $((\Sigma_{i-1}, \Gamma_{i-1}), M_{i-1})$ for a successor ordinal $i > 0$. By Definition 23, $(\Sigma_{i-1} \xrightarrow{v_{i-1,i}} \Sigma_i) = (\Sigma_{i-1} \xrightarrow{u_i} \mathcal{L}_i); (\mathcal{L}_i \xrightarrow{w_i} \Sigma_i)$ for some $u_i, w_i \in \mathcal{D}$ and $v_{i,\lambda}(w_i(e_i)) = \rho_i$ for some $e_i \in \mathbb{S}en(\mathcal{L}_i)$. There are two cases:

   (a) If there is no $u_i$-expansion $N_i$ of $M_{i-1}$ such that $N_i \models_{\mathcal{L}_i} e_i$ then we define $\Gamma_i = v_{i-1,i}(\Gamma_{i-1}) \cup \{\neg w_i(e_i)\}$. By Definition 23, there exists a $v_{i-1,i}$-expansion $M_i$ of $M_{i-1}$. Since $M_i\upharpoonright_{w_i}$ is a $u_i$-expansion of $M_{i-1}$, $M_i\upharpoonright_{w_i} \models_{\mathcal{L}_i} \neg e_i$, and by the satisfaction condition, $M_i \models_{\Sigma_i} \neg w_i(e_i)$. Using the satisfaction condition again, $M_i \models_{\Sigma_i} v_{i-1,i}(\Gamma_{i-1})$. It follows that $M_i \models_{\Sigma_i} \Gamma_i$.

[4] This condition means that no sentence of $I^0$ is constructed by applying Boolean connectives or/and first-order quantification.

(b) If there exists a $u_i$-expansion $N_i$ of $M_{i-1}$ such that $N_i \models_{\mathcal{L}_i} e_i$ then we have two subcases,

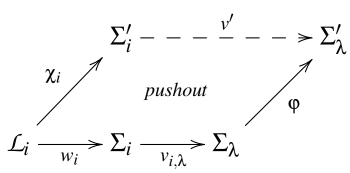
(i) $e_i$ is an existentially quantified sentence, i.e. $e_i = (\exists \chi_i)\delta_i$ for some $\mathcal{L}_i \xrightarrow{\chi_i} \Sigma'_i \in \mathcal{Q}$ and $\delta_i \in \mathbb{S}en(\Sigma'_i)$: There exists a $\chi_i$-expansion $N'_i$ of $N_i$ such that $N'_i \models_{\Sigma'_i} \delta_i$. Since $N'_i$ is an expansion of $M$ along $(v_{0,i-1}; u_i; \chi_i)$, by Definition 23, there exists $\psi_i : \Sigma'_i \to \Sigma_i$ conservative such that $\chi_i; \psi_i = w_i$. We define $\Gamma_i = v_{i-1,i}(\Gamma_{i-1}) \cup \{w_i(e_i), \psi_i(\delta_i)\}$. Let $M_i$ be a $\psi_i$-expansion of $N'_i$. Since $N'_i \models_{\Sigma'_i} \delta_i$, by the satisfaction condition, $M_i \models_{\Sigma_i} \psi_i(\delta_i)$. Since $M_i$ is a $w_i$-expansion of $N_i$, $M_i \models_{\Sigma_i} w_i(e_i)$. It follows that $M_i \models_{\Sigma_i} \Gamma_i$.

(ii) $e_i$ is not an existentially quantified sentence: We define the set of sentences $\Gamma_i = v_{i-1,i}(\Gamma_{i-1}) \cup \{w_i(e_i)\}$. Let $M_i$ be a $w_i$-expansion of $N_i$ and note that $M_i \models_{\Sigma_i} \Gamma_i$.

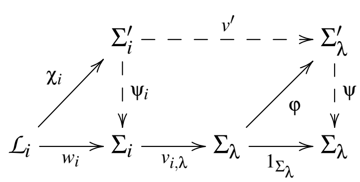
(2) Let $\tau \leq \lambda$ be a limit ordinal, and assume that we have defined $((\Sigma_i, \Gamma_i), M_i)$ for all ordinals $i < \tau$. Let $\Gamma_\tau = \cup_{i<\tau} v_{i,\tau}(\Gamma_i)$. Since $I$ is inductive-exact, there exists a $\Sigma_\tau$-model $M_\tau$ such that $M_\tau \upharpoonright_{v_{i,\tau}} = M_i$ for all $i < \tau$. For any $\gamma_\tau \in \Gamma_\tau$ there exists $j < \tau$ and $\gamma_j \in \Gamma_j$ such that $v_{j,\tau}(\gamma_j) = \gamma_\tau$. Since $M_j \models_{\Sigma_j} \Gamma_j$, $M_j \models_{\Sigma_j} \gamma_j$, and by the satisfaction condition, $M_\tau \models_{\Sigma_\tau} \gamma_\tau$.

(3) For all $\rho \in \mathbb{S}en(\Sigma_\lambda)$ we have $\rho \in \Gamma_\lambda$ or $\neg\rho \in \Gamma_\lambda$. Hence, $\Gamma_\lambda$ is maximal consistent.

We prove that for all $(\exists \varphi)\gamma \in \Gamma_\lambda$, where $\Sigma_\lambda \xrightarrow{\varphi} \Sigma'_\lambda \in \mathcal{Q}$, there exists a signature morphism $\psi : \Sigma'_\lambda \to \Sigma_\lambda$ such that $\varphi; \psi = 1_{\Sigma_\lambda}$ and $\psi(\gamma) \in \Gamma_\lambda$. By the definition of $((\Sigma_i, \Gamma_i), M_i)_{i \leq \lambda}$, there exists $i < \lambda$ such that $v_{i,\lambda}(w_i(e_i)) = (\exists \varphi)\gamma$. By condition 2 and Assumption 12, $e_i$ is an existentially quantified sentence. It follows that $e_i$ is of the form $(\exists \chi_i)\delta_i$. By Assumption 13, there exists a pushout




such that $v'(\delta_i) = \gamma$. By Definition 23, there exists $\psi_i : \Sigma'_i \to \Sigma_i$ such that $\chi_i; \psi_i = w_i$. Since $\{\Sigma'_i \xleftarrow{\chi_i} \mathcal{L}_i \xrightarrow{w_i; v_{i,\lambda}} \Sigma_\lambda, \Sigma'_i \xrightarrow{v'} \Sigma'_\lambda \xleftarrow{\varphi} \Sigma_\lambda\}$ is a pushout, there exists $\psi : \Sigma'_\lambda \to \Sigma_\lambda$ such that $\varphi; \psi = 1_{\Sigma_\lambda}$ and $v'; \psi = \psi_i; v_{i,\lambda}$.




Note that $\psi_i(\delta_i) \in \Gamma_i$, and we have $v_{i,\lambda}(\psi_i(\delta_i)) = \psi(v'(\delta_i)) = \psi(\gamma) \in \Gamma_\lambda$.

Let $\Gamma^0_\lambda = \Gamma_\lambda \cap \mathbb{S}en^0(\Sigma_\lambda)$ and $M_{\Gamma^0_\lambda}$ be a basic $\mathbb{S}$-reachable model of $\Gamma^0_\lambda$. We prove by induction on the structure of sentences that for each $\rho \in \mathbb{S}en(\Sigma_\lambda)$,

$$\rho \in \Gamma_\lambda \text{ iff } M_{\Gamma^0_\lambda} \models \rho$$

- $\rho \in \mathbb{S}en^0(\Sigma_\lambda)$ : We have $\rho \in \Gamma_\lambda$ iff $\Gamma^0_\lambda \models_{\Sigma_\lambda} \rho$ iff $M_{\Gamma^0_\lambda} \models \rho$.

- $\neg\rho \in \mathbb{S}en(\Sigma_\lambda)$ : By the induction hypothesis, $\rho \notin \Gamma_\lambda$ iff $M_{\Gamma^0_\lambda} \not\models \rho$. We have $\neg\rho \in \Gamma_\lambda$ iff $\rho \notin \Gamma_\lambda$ iff $M_{\Gamma^0_\lambda} \not\models \rho$ iff $M_{\Gamma^0_\lambda} \models \neg\rho$.

- $\vee E \in \mathbb{S}en(\Sigma_\lambda)$ : Since $\Gamma_\lambda$ is maximal consistent, $\vee E \in \Gamma_\lambda$ iff $\rho \in \Gamma_\lambda$ for some $\rho \in E$. By the induction hypothesis, $\rho \in \Gamma_\lambda$ for some $\rho \in E$ iff $M_{\Gamma^0_\lambda} \models \rho$ for some $\rho \in E$. By the definition of satisfaction, $M_{\Gamma^0_\lambda} \models \rho$ for some $\rho \in E$ iff $M_{\Gamma^0_\lambda} \models \vee E$.

- $(\exists\varphi)\gamma \in \mathbb{S}en(\Sigma_\lambda)$: Firstly, we prove $(\exists\varphi)\gamma \in \Gamma_\lambda$ implies $M_{\Gamma^0_\lambda} \models (\exists\varphi)\gamma$. If $(\exists\varphi)\gamma \in \Gamma_\lambda$, where $\Sigma_\lambda \xrightarrow{\varphi} \Sigma'_\lambda \in \mathcal{Q}$, then there exists $\psi : \Sigma'_\lambda \to \Sigma_\lambda$ such that $\varphi;\psi = 1_{\Sigma_\lambda}$ and $\psi(\rho) \in \Gamma_\lambda$. By Assumption 12, we can apply the induction hypothesis to $\psi(\rho)$, and we obtain $M_{\Gamma^0_\lambda} \models \psi(\rho)$. By the satisfaction condition, $M_{\Gamma^0_\lambda}\upharpoonright_\psi \models_{\Sigma'_\lambda} \rho$, since $M_{\Gamma^0_\lambda}\upharpoonright_\psi$ is a $\varphi$-expansion of $M_{\Gamma^0_\lambda}$, we get $M_{\Gamma^0_\lambda} \models (\exists\varphi)\gamma$.

  For the converse implication, we assume that $M_{\Gamma^0_\lambda} \models (\exists\varphi)\gamma$. Let $N$ be a $\varphi$-expansion of $M_{\Gamma^0_\lambda}$ such that $N \models \rho$. Since $M_{\Gamma^0_\lambda}$ is $\mathbb{S}$-reachable, there exists a substitution $\theta : \varphi \to 1_{\Sigma_\lambda} \in \mathbb{S}(\Sigma_\lambda)$ such that $M_{\Gamma^0_\lambda}\upharpoonright_\theta = N$. By the satisfaction condition, $M_{\Gamma^0_\lambda} \models \theta(\rho)$. By condition 5 and Assumption 29, we can apply the induction hypothesis to $\theta(\rho)$, and we obtain $\theta(\rho) \in \Gamma_\lambda$. It follows that $\Gamma_\lambda \cup \{(\exists\varphi)\gamma\}$ is consistent, and since $\Gamma_\lambda$ is maximal consistent, $(\exists\varphi)\gamma \in \Gamma_\lambda$.

□

Theorem 46 can be applied to **FOL** and **POA**. We will focus on **FOL** since the other case is similar.

**Corollary 47** **FOL** has the $\mathbb{S}^{\mathbf{FOL}}$-DLSP.

PROOF. We set the parameters of Theorem 46. The institution $I$ is **FOL**, and the institution $I^0$ is $\mathbf{FOL}^0$, the restriction of **FOL** to atomic sentences. $\mathcal{Q}$ is $\mathcal{Q}^{\mathbf{FOL}}$, the broad subcategory of signature extensions with a finite number of variables.

Notice that no sentence in $\mathbf{FOL}^0$ is obtained by applying Boolean connectives or/and quantification. It follows that the first condition of Theorem 46 holds in **FOL**. By Proposition 17, **FOL** is exact, and in particular, **FOL** is inductive-exact. By Lemma 24, any signature $\Sigma$ has a $\mathcal{D}^{\mathbf{FOL}}$-extension via any $\Sigma$-model. By Lemma 15, any set of sentences in $\mathbf{FOL}^0$ is basic, and by Corollary 36, the basic models are $\mathbb{S}^{\mathbf{FOL}}$-reachable. By Theorem 46, **FOL** has $\mathbb{S}^{\mathbf{FOL}}$-DLSP. □

Our approach is general and may produce different results within a logical framework.

**Corollary 48** In **FOL**, any consistent set of quantifier-free sentences has a model which consists of interpretations of terms.

PROOF. We set the parameters of Theorem 46. The institution $I$ is the $\mathbf{FOL}^1$, the restriction of **FOL** to quantifier-free sentences, and the institution $I^0$ is $\mathbf{FOL}^0$, the restriction of **FOL** to atomic sentences. $\mathcal{Q}$ is $\mathcal{Q}^{\mathbf{FOL}^1}$, the broad subcategory of signature identities. $\mathcal{D}$ is $\mathcal{D}^{\mathbf{FOL}}$ and $\mathbb{S}$ is $\mathbb{S}^{\mathbf{FOL}^1}$, the restriction of $\mathbb{S}^{\mathbf{FOL}}$ to quantifier-free sentences (i.e. for all $\Sigma \in |\mathbb{S}ig^{\mathbf{FOL}}|$ and $\theta^1 \in \mathbb{S}^{\mathbf{FOL}^1}(\Sigma)$ there exists $\theta \in \mathbb{S}^{\mathbf{FOL}}(\Sigma)$ such that $\mathbb{S}en(\theta^1)$ is the restriction of $\mathbb{S}en(\theta)$ to quantifier-free sentences and $\mathbb{M}od(\theta^1) = \mathbb{M}od(\theta)$).

Let $\Sigma \in |\mathbb{S}ig^{\mathbf{FOL}}|$ be a signature and $\Gamma \subseteq \mathbb{S}en^{\mathbf{FOL}^1}(\Sigma)$ a consistent set of quantifier-free sentences. It follows that there exists a $\Sigma$-model $M$ satisfying $\Gamma$. It is straightforward to check that $1_\Sigma$ is a $\mathcal{D}^{\mathbf{FOL}}$-extension via $M$ (in $\mathbf{FOL}^1$). By Theorem 46, there exists a $\Sigma$-morphism $h : N \to M$ that preserves the satisfaction of sentences such that $N$ is $\mathbb{S}^{\mathbf{FOL}}$-reachable. By Proposition 35, $N$ consists of interpretations of terms. Since $\mathbf{FOL}^1$ has negations, $N \models_\Sigma \Gamma$. $\square$

The result provided by Theorem 46 is very general but it has some limitations. For example, it cannot be applied to **HNK** because the sets of atoms are not basic. In **PA**, the basic models corresponding to atomic sentences are not reachable by the total operation symbols. It follows that partial algebra doesn't fall into the framework of Theorem 46.

# 4 Borrowing Downward Löwenheim-Skolem Property

In this section we develop a second method for proving the DLSP, by borrowing it across institution mappings for more expressive logical systems which are encoded into the institution of presentations of less refined institutions. The institution mappings used in this section for borrowing results are institution comorphisms [20].

**Definition 49** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ and $I' = (\mathbb{S}ig', \mathbb{S}en', \mathbb{M}od', \models')$ be two institutions. An *institution comorphism* $(\phi, \alpha, \beta) : I \to I'$ consists of

- a functor $\phi : \mathbb{S}ig \to \mathbb{S}ig'$, and
- two natural transformations $\alpha : \mathbb{S}en \Rightarrow \phi; \mathbb{S}en'$ and $\beta : \phi^{op}; \mathbb{M}od' \Rightarrow \mathbb{M}od$ such that the following satisfaction condition for institution comorphisms holds:

$$M' \models'_{\phi(\Sigma)} \alpha_\Sigma(e) \text{ iff } \beta_\Sigma(M') \models_\Sigma e$$

  for every signature $\Sigma \in |\mathbb{S}ig|$, each $\phi(\Sigma)$-model $M'$, and any $\Sigma$-sentence $e$.

We say that $\beta_\Sigma$ is *conservative*, where $\Sigma \in |\mathbb{S}ig|$, if for all $\Sigma$-models $M$ there exists a $\phi(\Sigma)$-model $M'$ such that $\beta_\Sigma(M') = M$.

The central result of this section consists of borrowing the DLSP across an institution comorphism.

**Theorem 50** Let $(\phi, \alpha, \beta) : I \to I'$ be a comorphism of institutions such that for all signatures $\Sigma \in |\mathbb{S}ig|$,

1. $\beta_\Sigma$ is conservative, and
2. $card(\mathbb{S}en(\Sigma)) = card(\mathbb{S}en'(\phi(\Sigma)))$.

Consider two broad subcategories $\mathcal{D} \subseteq \mathbb{S}ig$ and $\mathcal{D}' \subseteq \mathbb{S}ig'$ such that

3. for all signatures $\Sigma \in |\mathbb{S}ig|$ and $\mathcal{D}'$-chains $v' : \phi(\Sigma) \to \Sigma'_1$ there exists a $\mathcal{D}$-chain $v : \Sigma \to \Sigma_1$ such that $\phi(v) = v'$.

Assume two substitution functors $\mathbb{S} : \mathcal{D}^{op} \to \mathbb{CAT}$ (for the institution $I$) and $\mathbb{S}' : \mathcal{D}'^{op} \to \mathbb{CAT}$ (for the institution $I'$) such that

4. for all models $M' \in |\mathbb{M}od'(\phi(\Sigma))|$, if $M'$ is $\mathbb{S}'$-reachable then $\beta_\Sigma(M')$ is $\mathbb{S}$-reachable.

If $I'$ has the $\mathbb{S}'$-DLSP then $I$ has the $\mathbb{S}$-DLSP.

PROOF. Assume $I'$ has the $\mathbb{S}'$-DLSP, and let $\Sigma \in |\mathbb{S}ig|$ and $M \in |\mathbb{M}od(\Sigma)|$. Since $\beta_\Sigma$ is conservative, there exists $M' \in |\mathbb{M}od'(\phi(\Sigma))|$ such that $\beta_\Sigma(M') = M$. Since $I'$ has $\mathbb{S}'$-DLSP, there exists

- a $\mathcal{D}'$-chain $v' : \phi(\Sigma) \to \Sigma'_1$ such that $\Sigma'_1$ has the same power as $\phi(\Sigma)$, and
- a $\Sigma'_1$-morphism $h'_1 : N'_1 \to M'_1$ such that $N'_1$ is $\mathbb{S}'$-reachable, $M'_1\upharpoonright_{v'} = M'$ and $h'_1$ preserves satisfaction of sentences.

By condition 3, there exists a $\mathcal{D}$-chain $\Sigma \xrightarrow{v} \Sigma_1$ such that $\phi(\Sigma \xrightarrow{v} \Sigma_1) = (\phi(\Sigma) \xrightarrow{v'} \Sigma'_1)$. By condition 2, $card(\mathbb{S}en(\Sigma)) = card(\mathbb{S}en'(\phi(\Sigma))$ and $card(\mathbb{S}en(\Sigma_1)) = card(\mathbb{S}en'(\Sigma'_1))$, which implies that $\Sigma$ and $\Sigma_1$ have the same power. Since $N'_1$ is $\mathbb{S}'$-reachable, by condition 4, $\beta_{\Sigma_1}(N'_1)$ is $\mathbb{S}$-reachable. Notice that $\beta_{\Sigma_1}(M'_1)\upharpoonright_v = \beta_\Sigma(M'_1\upharpoonright_{v'}) = \beta_\Sigma(M') = M$. By the satisfaction condition for institution comorphisms, $\beta_{\Sigma_1}(N'_1 \xrightarrow{h'_1} M'_1)$ preserves satisfaction of sentences: indeed, for all $\rho \in \mathbb{S}en(\Sigma_1)$, if $\beta_{\Sigma_1}(N'_1) \models_{\Sigma_1} \rho$ then $N'_1 \models_{\Sigma'_1} \alpha_{\Sigma_1}(\rho)$, and since $h'_1$ preserves satisfaction of sentences, $M'_1 \models_{\Sigma'_1} \alpha_{\Sigma_1}(\rho)$, which implies $\beta_{\Sigma_1}(M'_1) \models_{\Sigma_1} \rho$. It follows that $I$ has the $\mathbb{S}$-DLSP. $\square$

In our concrete examples $I'$ is the institution of presentations over a base institution which has DLSP. It is desirable to lift DLSP from a base institution to the institution of its presentations.

**Proposition 51** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution with negations, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S} : \mathcal{D}^{op} \to \mathbb{CAT}$ a substitution functor such that $I$ has $\mathbb{S}$-DLSP. Then $I^{pres}$ has $\mathbb{S}$-DLSP.

PROOF. Let $(\Sigma, E)$ be a presentation and $M \in |\mathbb{M}od(\Sigma, E)|$ a model. Since $I$ has $\mathbb{S}$-DLSP, there exists

- a $\mathcal{D}$-chain $v : \Sigma \to \Sigma'$ such that the power of $\Sigma'$ is equal to the power of $\Sigma$, and
- a $\Sigma'$-morphism $h' : N' \to M'$ such that $N'$ is $\mathbb{S}^I$-reachable, $M'\upharpoonright_v = M$ and $h'$ preserves satisfaction of sentences.

It follows that $v : (\Sigma, E) \to (\Sigma', v(E))$ is a $\mathcal{D}$-chain in $I^{pres}$. By the satisfaction condition, $M' \models_{\Sigma'} v(E)$. Since $I$ has negations and $h'$ preserves satisfaction of sentences, $N' \models_{\Sigma'} v(E)$ meaning that $N' \xrightarrow{h'} M' \in \mathbb{M}od(\Sigma_\lambda, v(E))$. By Proposition 40, $N'$ is $\mathbb{S}$-reachable in $I^{pres}$. Hence, $I^{pres}$ has the $\mathbb{S}$-DLSP. $\square$

## 4.1 Downward Löwenheim-Skolem Property in Partial Algebra

In order to establish that **PA** has $\mathbb{S}^{\mathbf{PA}}$-DLSP we set the parameters of Theorem 50. We recall the definition of a comorphism

$$(\phi, \alpha, \beta) : \mathbf{PA} \to \mathbf{FOL}^{pres}$$

which can be found, for example, in [33] or [32].

- Each **PA**-signature $(S,TF,PF)$ is mapped to the first-order presentation $((S,TF,\overline{PF}),E_{(S,TF,PF)})$, where $\overline{PF}_{ws}=PF_{w\to s}$ for all $(w,s)\in S^*\times S$, and $E_{(S,TF,PF)}=\{(\forall X\cup\{y,z\})\sigma(X,y)\wedge\sigma(X,z)\Rightarrow(y=z)\mid\sigma\in PF\}$.
- For all **PA** signatures $(S,TF,PF)$ we have
  (1) $\alpha_{(S,TF,PF)}(t\stackrel{e}{=}t')=(\exists X\cup\{x\})bind(t,x)\wedge bind(t',x)$, where
    (a) for each term $t\in T_{(S,TF\cup PF)}$ and variable $x$, $bind(t,x)$ is a finite conjunction of atoms defined as follows: $bind(\sigma(t_1,\dots,t_n),x)=\bigwedge\limits_{1\leq i\leq n}bind(t_i,x_{t_i})\wedge$
    $\begin{cases}\sigma(x_{t_1},\dots,x_{t_n})=x & \text{if }\sigma\in TF\\ \sigma(x_{t_1},\dots,x_{t_n},x) & \text{if }\sigma\in PF\end{cases}$
    (b) $X$ is the set of variables introduced by $bind(t,x)$ and $bind(t',x)$.
  (2) $\alpha_{(S,TF,PF)}$ commutes with the first-order constructors for sentences.
- For each $((S,TF,\overline{PF}),E_{(S,TF,PF)})$-model $M$, the algebra $\beta_{(S,TF,PF)}(M)$ is defined as follows:
  (1) $\beta_{(S,TF,PF)}(M)_x=M_x$ for all $x\in S$ or $x\in TF$,
  (2) $\beta_{(S,TF,PF)}(M)_\sigma(m)=n$ for all $\sigma\in PF$ such that $(m,n)\in M_\sigma$.

**Lemma 52** For all $(S,TF,PF)\in|\mathbb{S}ig^{\mathbf{PA}}|$ and $M\in|\mathbb{M}od^{\mathbf{FOL}}((S,TF,\overline{PF}),E_{(S,TF,PF)})|$, we have that $\beta_{(S,TF,PF)}(M)$ is $\mathbb{S}^{\mathbf{PA}}$-reachable in **PA** if $M$ is $\mathbb{S}^{\mathbf{FOL}}$-reachable in $\mathbf{FOL}^{pres}$.

PROOF. By Proposition 35 and Proposition 40, in $\mathbf{FOL}^{pres}$, any model which consists of interpretation of terms is $\mathbb{S}^{\mathbf{FOL}^{pres}}$-reachable. One can easily prove the converse implication by repeating the argument used in the second part of the proof of Proposition 35. In particular, for any signature $(S,TF,PF)\in|\mathbb{S}ig^{\mathbf{PA}}|$, each $\mathbb{S}^{\mathbf{FOL}^{pres}}$-reachable $((S,TF,\overline{PF}),E_{(S,TF,PF)})$-model $M$ consists of interpretations of $(S,TF)$-terms. It follows that $\beta_{(S,TF,PF)}(M)$ consists of interpretations of $(S,TF)$-terms. By Proposition 38, $\beta_{(S,TF,PF)}(M)$ is $\mathbb{S}^{\mathbf{PA}}$-reachable. □

**Corollary 53** **PA** has the $\mathbb{S}^{\mathbf{PA}}$-DLSP.

PROOF. By Proposition 51, we lift the $\mathbb{S}^{\mathbf{FOL}}$-DLSP from the institution **FOL** to $\mathbf{FOL}^{pres}$. Then we apply Theorem 50 to the comorphism above. Note that for all **PA** signatures $(S,TF,PF)$ we have:

1. $\beta_{(S,TF,PF)}$ is conservative because it is an isomorphism.
2. For all signature extensions with constants $v':(S,TF,\overline{PF}),E_{(S,TF,PF)})\hookrightarrow(S,TF\cup C,\overline{PF}),E_{(S,TF,PF)})\in|\mathbb{S}ig^{\mathbf{FOL}^{pres}}|$ we define the signature extension with total constants $v:(S,TF,PF)\hookrightarrow(S,TF\cup C,PF)\in|\mathbb{S}ig^{\mathbf{PA}}|$ and we get $\phi(v)=v'$.
3. By Lemma 52, for all $((S,TF,\overline{PF}),E_{(S,TF,PF)})$-models $M$, $\beta_{(S,TF,PF)}(M)$ is $\mathbb{S}^{\mathbf{PA}}$-reachable if $M$ is $\mathbb{S}^{\mathbf{FOL}}$-reachable.

Therefore the conditions of Theorem 50 are fulfilled and we conclude that **PA** has $\mathbb{S}^{\mathbf{PA}}$-DLSP. □

## 4.2 Downward Löwenheim-Skolem Property in Higher-Order Logic

We will show that $\mathbf{HNK}^i$ falls into the framework of Theorem 46. Then we borrow the DLSP from $\mathbf{HNK}^i$ for $\mathbf{HNK}$ along an institution comorphism.

**Lemma 54** In $\mathbf{HNK}^i$, any set of atomic sentences is epi basic.

PROOF. Let $(S,F)$ be a $\mathbf{HNK}$ signature and $B \subseteq \mathbb{S}en^{\mathbf{HNK}}(S,F)$ a set of equational atoms. Let $T_{(S,F)} \in |\mathbb{M}od^{\mathbf{HNK}^i}(S,F)|$ be the term model such that for all types $s_1,s_2 \in \overrightarrow{S}$ and terms $t \in (T_{(S,F)})_{s_1 \to s_2}$, the intensional function $t : (T_{(S,F)})_{s_1} \to (T_{(S,F)})_{s_2}$ is defined by $t(x) = tx$ for all $x \in (T_{(S,F)})_{s_1}$. Let $\equiv_B$ be the $\mathbf{HNK}^i$ congruence on $T_{(S,F)}$ generated by the equational atoms in $B$, i.e. the least equivalence relation on $T_{(S,F)}$ closed to the following properties: $\dfrac{t_1 = t_2 \in B}{t_1 \equiv_B t_2}$, $\dfrac{t_1 \equiv_B t_2}{tt_1 \equiv_B tt_2}$, and $\dfrac{t_1 \equiv_B t_2 \in B}{t_1 t \equiv_B t_2 t}$, where $t, t_1, t_2$ are terms of appropriate types. The basic model $M_B$ is the quotient $T_{(S,F)}/_{\equiv_B}$ of the term model $T_{(S,F)}$ by the congruence $\equiv_B$. It is straightforward to show that for any model $M \in |\mathbb{M}od^{\mathbf{HNK}^i}(S,F)|$ we have that $M \models_{(S,F)} B$ iff there exists a unique arrow $M_B \to M$. □

Let $\mathbb{S}^{\mathbf{HNK}^i} : (\mathcal{D}^{\mathbf{HNK}})^{op} \to \mathbb{CAT}$ denote the sub-functor of $\mathbb{S}ub^{\mathbf{HNK}^i} : (\mathbb{S}ig^{\mathbf{HNK}})^{op} \to \mathbb{CAT}$ which maps each signature $(S,F)$ to the subcategory of $(S,F)$-substitutions represented by functions $\theta : C_1 \to T_{(S,F)}(C_2)$, where $C_1$ and $C_2$ are finite sets of function symbols (of any type) different from the symbols in $F$. In $\mathbf{HNK}^i$, $\mathbb{S}^{\mathbf{HNK}^i}$-reachable models consists of interpretation of terms.

**Proposition 55** A $\mathbf{HNK}^i$ model is $\mathbb{S}^{\mathbf{HNK}^i}$-reachable iff its elements consists of interpretations of terms.

PROOF. Similar to the proof of Proposition 39. □

The following result is a consequence of Theorem 46.

**Corollary 56** $\mathbf{HNK}^i$ has $\mathbb{S}^{\mathbf{HNK}^i}$-DLSP.

PROOF. We set the parameters of Theorem 46. The institution $I$ is $\mathbf{HNK}^i$ and $I^0$ is $(\mathbf{HNK}^i)^0$, the restriction of $\mathbf{HNK}^i$ to atomic sentences. $\mathcal{Q}$ is $\mathcal{Q}^{\mathbf{HNK}}$, the broad subcategory of signature extensions with a finite number of variables of any type.

Given a signature $(S,F) \in |\mathbb{S}ig^{\mathbf{HNK}}|$ and a model $M \in |\mathbb{M}od^{\mathbf{HNK}^i}(S,F)|$, let $\lambda$ be the power of $(S,F)$ and let $C$ be a set of function symbols different from the symbols in $F$ such that for all types $s \in \overrightarrow{S}$ we have $card(C_s) = \lambda$ if $M_s \neq \emptyset$, and $C_s = \emptyset$ if $M_s = \emptyset$. By the same argument used in case of **FOL** (see Lemma 24), the inclusion $(S,F) \hookrightarrow (S, F \cup C)$ is a $\mathcal{D}^{\mathbf{HNK}}$-extension of $(S,F)$ via $M$. It follows that all conditions of Theorem 46 hold in $\mathbf{HNK}^i$. Hence, $\mathbf{HNK}^i$ has $\mathbb{S}^{\mathbf{HNK}}$-DLSP. □

We borrow the DLSP from $(\mathbf{HNK}^i)^{pres}$ for $\mathbf{HNK}$ across the comorphism $(\phi_e, \alpha_e, \beta_e)$ : $\mathbf{HNK} \to (\mathbf{HNK}^i)^{pres}$ which is defined as follows:

- each $\mathbf{HNK}$ signature $\Sigma = (S,F)$ is mapped to the presentation $(\Sigma, E_\Sigma)$, where $E_\Sigma = \{(\forall f,g)((\forall a) fa = ga) \Rightarrow (f = g) \mid f = (x, s_1 \to s_2, \Sigma),\ g = (y, s_1 \to s_2, \Sigma), a = (z, s_1, \Sigma), s_i \in \overrightarrow{S}\}$ and $x$, $y$, $z$ are (fixed) variable names,

- $\alpha_e$ is the identity natural transformation,
- for each **HNK** signature $\Sigma$, $(\beta_e)_\Sigma : \mathbb{M}od^{(\mathbf{HNK}^i)^{pres}}(\Sigma, E_\Sigma) \to \mathbb{M}od^{\mathbf{HNK}}(\Sigma)$ is the forgetful functor which drops the name of intensional functions.

The following result is obtained by applying Theorem 50 to the comorphism $(\phi_e, \alpha_e, \beta_e) : \mathbf{HNK} \to (\mathbf{HNK}^i)^{pres}$.

**Corollary 57** **HNK** has the $\mathbb{S}^{\mathbf{HNK}}$-DLSP.

Proof. It is straightforward to check that the comorphism $(\phi_e, \alpha_e, \beta_e) : \mathbf{HNK} \to (\mathbf{HNK}^i)^{pres}$ satisfies the conditions of Theorem 50. □

# 5 Borrowing Interpolation

In this section we use Downward Löwenheim-Skolem Theorem to carry out the interpolation from a base institution to its constructor-based counterpart across forgetful institution morphisms. Below we recall the concept of Craig interpolation and other necessary notions for developing our results.

## 5.1 Preliminaries

Let $\varphi : (S, F, F^c, P) \to (S', F', F'^c, P')$ be a signature morphism in **CFOL**. We say that $\varphi^{op}$ is *injective* if for all arities $w \in S^*$ and sorts $s \in S$, $\varphi^{op}_{w\to s}$ is injective. The same applies to $\varphi^{ct}$, the constructors component, and $\varphi^{rl}$, the relations component. $\varphi^{op}$ is *encapsulated* means that no "new" operation symbol, i.e. outside the image of $\varphi$, is allowed to have the sort in the image of $\varphi$. More precise, if $\sigma' \in F'_{w'\to s'}$ then for all $s \in S$ such that $s' = \varphi^{st}(s)$ there exists $\sigma \in F_{w\to s}$ such that $\varphi^{op}(\sigma) = \sigma'$. The same applies to $\varphi^{ct}$.

**Definition 58 (($xyzt$)-signature morphisms)** A **CFOL** signature morphism $\varphi : (S, F, F^c, P) \to (S', F', F'^c, P')$ is a $(xyzt)$-morphism, with $x, t \in \{i, *\}$ and $y, z \in \{i, e, *\}$, where $i$ stands for "injective", $e$ for "encapsulated", and $*$ for "all", when

(1) the sort component $\varphi^{st} : S \to S'$ has the property $x$,

(2) the operation component $\varphi^{op} = (\varphi^{op}_{w\to s} : F_{w\to s} \to F'_{\varphi^{st}(w)\to\varphi^{st}(s)})_{(w,s)\in S^*\times S}$ has the property $y$,

(3) the constructor component $\varphi^{ct} = (\varphi^{ct}_{w\to s} : F^c_{w\to s} \to F'^c_{\varphi^{st}(w)\to\varphi^{st}(s)})_{(w,s)\in S^*\times S}$ has the property $z$, and

(4) the relation component $\varphi^{rl} = (\varphi^{rl}_w : P_w \to P'_{\varphi^{st}(w)})_{w\in S^*}$ has the property $t$.

This notational convention can be extended to other institutions too, such as for example **FOL**. In case of **FOL**, because we do not have constructor symbols the third component is missing.

The category of **CFOL** signature morphisms does not have pushouts, in general. However, if we restrict the class of arrows then pushouts may exist. Below we recall some results from the literature.

**Proposition 59** [23] In **CFOL**, the subcategory of $(i*e*)$-signature morphisms is strongly closed under pushouts.

**Proposition 60** [2] In **CFOL**, the subcategory of $(**e*)$-signature morphisms has pushouts.

Our interpolation results are closely linked to the notion of sufficient completeness.

**Definition 61** A **CFOL** presentation $((S,F,F^c,P),E)$ is *sufficient complete* when for all $(S,F,P)$-models $M$, if $M \in |\mathbb{M}od^{\mathbf{CFOL}}((S,F,F^{S^c},P),E)|$ then $M \in |\mathbb{M}od^{\mathbf{CFOL}}((S,F,F^c,P),E)|$.

In other words, a presentation $((S,F,F^c,P),E)$ is sufficient complete when for all $(S,F,P)$-models $M$, if $M$ is reachable by the operations in $F^{S^c}$ and satisfies $E$ then $M$ is reachable by the constructors in $F^c$.

Let $\mathbb{S}ig^{\mathbf{CFOL}^{sc}} \subseteq \mathbb{S}ig^{\mathbf{CFOL}^{pres}}$ be the full subcategory of sufficient complete presentations. We define the *institution of sufficient complete presentations* $\mathbf{CFOL}^{sc}$ as the restriction of $\mathbf{CFOL}^{pres}$ to the sufficient complete presentations.

**Proposition 62** [21] The inclusion functor $\mathbb{S}ig^{\mathbf{CFOL}^{sc}} \hookrightarrow \mathbb{S}ig^{\mathbf{CFOL}^{pres}}$ lifts the $((iee*)^{pres},(****)^{pres})$-pushouts.

**Proposition 63** [21] The inclusion functor $\mathbb{S}ig^{\mathbf{CFOL}^{sc}} \hookrightarrow \mathbb{S}ig^{\mathbf{CFOL}^{pres}}$ lifts the $((**e*)^{pres},(**e*)^{pres})$-pushouts.

Below we recall the institution-independent concept of Craig interpolation.

**Definition 64 (Craig Interpolation)** In any institution a commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\varphi_2} & \Sigma \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle\chi_1} \\ \Sigma_0 & \xrightarrow[\varphi]{} & \Sigma_1 \end{array}$$

is a *Craig Interpolation square (CI square)* iff for each set $E_1$ of $\Sigma_1$-sentences and any set $E_2$ of $\Sigma_2$-sentences such that $\chi_1(E_1) \models \varphi_2(E_2)$ there exists $E_0 \subseteq \mathbb{S}en(\Sigma_0)$ such that $E_1 \models \varphi(E_0)$ and $\chi(E_0) \models E_2$.

The Craig interpolation property can be strengthened by adding to the initial premises $E_1$ a set $\Gamma_2$ of $\Sigma_2$-sentences as secondary premises.

**Definition 65 (Craig-Robinson Interpolation)** In any institution we say that a commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\varphi_2} & \Sigma \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle\chi_1} \\ \Sigma_0 & \xrightarrow[\varphi]{} & \Sigma_1 \end{array}$$

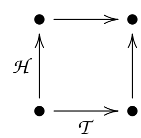
is a *Craig-Robinson Interpolation square (CRI square)* iff for each set $E_1$ of $\Sigma_1$-sentences and any sets $E_2$ and $\Gamma_2$ of $\Sigma_2$-sentences such that $\chi_1(E_1) \cup \varphi_2(\Gamma_2) \models \varphi_2(E_2)$ there exists a set $E_0$ of $\Sigma_0$-sentences such that $E_1 \models \varphi(E_0)$ and $\chi(E_0) \cup \Gamma_2 \models E_2$.

The name "Craig-Robinson" interpolation has been used for instances of this property in [37, 17]. Note that a CRI square is also a CI square, and under certain conditions, such as compactness and the presence of implications, CI is equivalent to CRI. For a proof of the following lemma see for example [12].

**Lemma 66** In any compact institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ with implications a pushout square of signature morphisms is a CI square iff it is a CRI square.

The above lemma does not hold in institutions that do not have implications such as **UnivFOL**.

**Definition 67** An institution has *Craig* $(\mathcal{T},\mathcal{H})$*-interpolation* ($(\mathcal{T},\mathcal{H})$*-CI*), respectively, *Craig-Robinson* $(\mathcal{T},\mathcal{H})$*-interpolation* ($(\mathcal{T},\mathcal{H})$*-CRI*), for two subcategories of signature morphisms $\mathcal{T}$ and $\mathcal{H}$ if each pushout square of signature morphisms of the form




is a CI square, respectively, CRI square.

Below there are some interpolation results from the literature.

**Remark 68** According to [12], **UnivFOL** has

1. $((ie*),(***))$-CRI, and
2. $((***),(iii))$-CI.

We borrow interpolation from a base institution for its constructor-based version across a forgetful institution morphism. Institution morphisms were introduced in [18] and are suitable to formalise forgetful mappings between more complex institutions to simpler ones.

**Definition 69** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ and $I' = (\mathbb{S}ig', \mathbb{S}en', \mathbb{M}od', \models')$ be two institutions. An *institution morphism* $(\phi, \alpha, \beta) : I \to I'$ consists of

- a functor $\phi : \mathbb{S}ig \to \mathbb{S}ig'$, and
- two natural transformations $\alpha : \phi;\mathbb{S}en' \Rightarrow \mathbb{S}en$ and $\beta : \mathbb{M}od \Rightarrow \phi;\mathbb{M}od'$ such that the following satisfaction condition for institution morphisms holds:

$$M \models_\Sigma \alpha_\Sigma(e') \text{ iff } \beta_\Sigma(M) \models'_{\phi(\Sigma)} e'$$

  for all signatures $\Sigma \in |\mathbb{S}ig|$, $\Sigma$-models $M$, and $\phi(\Sigma)$-sentences $e'$.

We define the institution morphism $\Delta_{\mathbf{UnivCFOL}} = (\phi, \alpha, \beta) : \mathbf{UnivCFOL} \to \mathbf{UnivFOL}$ as follows:

(1) The functor $\phi$ maps every **UnivCFOL** signature morphism $(S,F,F^c,P) \xrightarrow{\varphi} (S',F',F'^c,P')$ to the **UnivFOL** signature morphism $(S,F,P) \xrightarrow{\varphi} (S',F',P')$.

(2) $\alpha$ is the identity natural transformation, i.e. for every **UnivCFOL** signature $(S,F,F^c,P)$ we have $\alpha_{(S,F,F^c,P)} = 1_{\mathbb{S}en(S,F,P)}$.

(3) $\beta$ is the inclusion natural transformation, i.e. for every **UnivCFOL** signature $(S,F,F^c,P)$, $\beta_{(S,F,F^c,P)} : \mathbb{M}od(S,F,F^c,P) \hookrightarrow \mathbb{M}od(S,F,P)$ is the inclusion functor.

## 5.2 Borrowing Result

The following result is based on Theorem 46 and it is the key for borrowing interpolation from a base institution with universally quantified sentences for its constructor-based version.

**Theorem 70** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{Q} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S}en^0 \subseteq \mathbb{S}en$ a sub-functor such that

1. all sentences of $I$ are of the form $(\forall\chi)\rho$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{Q}$ and $\rho \in \mathbb{S}en(\Sigma')$ is constructed from the sentences of $I^0 = (\mathbb{S}ig, \mathbb{S}en^0, \mathbb{M}od, \models)$ by applying Boolean connectives, and

2. any sentence of the form $(\forall\chi)\rho$ as above is a sentence of $I$.

Assume a subcategory $\mathcal{D}$ of signature morphisms such that $\mathcal{Q} \subseteq \mathcal{D} \subseteq \mathbb{S}ig$, and a substitution functor $\mathbb{S} : \mathcal{D}^{op} \to \mathbb{CAT}$ such that

3. all sentences of $I^0$ are basic and the basic models are $\mathbb{S}$-reachable.

Then for every signature $\Sigma \in |\mathbb{S}ig|$, each set of sentences $\Gamma \subseteq \mathbb{S}en(\Sigma)$, and any sentence $(\forall\chi)\rho \in \mathbb{S}en(\Sigma)$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{Q}$, we have

$$\Gamma \models_\Sigma (\forall\chi)\rho \text{ iff } M \models_{\Sigma'} \chi(\Gamma) \text{ implies } M \models \rho \text{ for all } \mathbb{S}\text{-reachable } \Sigma'\text{-models } M.$$

PROOF. The direct implication is obvious. For the converse implication, we show that if $\Gamma \not\models_\Sigma (\forall\chi)\rho$ then there exists a $\mathbb{S}$-reachable $\Sigma'$-model $M$ such that $M \models_{\Sigma'} \chi(\Gamma)$ and $M \not\models_{\Sigma'} \rho$.

Assume that $\Gamma \not\models_\Sigma (\forall\chi)\rho$. We define the set of $\Sigma'$-sentences $\Gamma^1 = \{\theta(\gamma) \mid (\forall\varphi)\gamma \in \chi(\Gamma),\ \varphi \xrightarrow{\theta} 1_{\Sigma'} \in \mathbb{S}(\Sigma')\}$. We have $\Gamma^1 \not\models_{\Sigma_1} \rho$ (if $\Gamma^1 \models_{\Sigma_1} \rho$ then since $\chi(\Gamma) \models_{\Sigma'} \Gamma^1$, we have $\chi(\Gamma) \models_{\Sigma'} \rho$, which contradicts our assumption). It follows that $\Gamma^1 \cup \{\neg\rho\}$ is consistent. Let $\mathbb{S}en^1$ be the sub-functor of $\mathbb{S}en$ which associates to each signature the set of sentences obtained from the sentences of $I^0$ by applying Boolean connectives. We apply Theorem 46 to the institution $I^1 = (\mathbb{S}ig, \mathbb{S}en^1, \mathbb{M}od, \models)$ with the subcategory of signature morphisms used for quantification consisting of identities. Let $N \in |\mathbb{M}od(\Sigma', \Gamma^1 \cup \{\neg\rho\})|$, and note that the identity $1_{\Sigma'}$ is a $\mathcal{D}$-extension via $N$ in $I^1$. By Theorem 46, there exists a $\Sigma'$-morphism $h : M \to N$ such that $M$ is $\mathbb{S}$-reachable and $h$ preserves satisfaction of sentences. Since $N \models_{\Sigma'} \Gamma^1 \cup \{\neg\rho\}$ and $I^1$ has negations, we have $M \models_{\Sigma'} \Gamma^1 \cup \{\neg\rho\}$. We show that $M \models_{\Sigma'} \chi(\Gamma)$. Let $(\forall\varphi)\gamma \in \chi(\Gamma)$ with $\Sigma' \xrightarrow{\varphi} \Sigma'' \in \mathcal{Q} \subseteq \mathcal{D}$, and $M'$ a $\varphi$-expansion of $M$. Since $M$ is $\mathbb{S}$-reachable and $\varphi \in \mathcal{D}$, there exists a substitution $\theta' : \varphi \to 1_{\Sigma'}$ such that $M\upharpoonright_{\theta'} = M'$. By the definition of $\Gamma^1$, we have $\theta'(\gamma) \in \Gamma^1$. Since $M \models_{\Sigma'} \Gamma^1$, we have $M \models_{\Sigma'} \theta'(\gamma)$, and by the satisfaction condition for substitutions, $M' \models_{\Sigma''} \gamma$. Hence, $M$ is $\mathbb{S}$-reachable, $M \models_{\Sigma'} \chi(\Gamma)$ and $M \not\models_{\Sigma'} \rho$. □

One interesting consequence of Theorem 70 is the following result.

**Corollary 71** For all $((S,F,F^c,P),\Gamma) \in |\mathbb{S}ig^{\mathbf{UnivCFOL}^{sc}}|$ and any $(\forall Y)\rho \in \mathbb{S}en(S,F,F^c,P)$ such all sorts of the variables in $Y$ are loose and $\rho$ is a quantifier-free $(S, F \cup Y, F^c, P)$-sentence, we have

$$\Gamma \models^{\mathbf{UnivFOL}}_{(S,F,P)} (\forall Y)\rho \text{ iff } \Gamma \models^{\mathbf{UnivCFOL}}_{(S,F,F^c,P)} (\forall Y)\rho$$

PROOF. It is obvious that $\Gamma \models^{\mathbf{UnivFOL}}_{(S,F,P)} (\forall Y)\rho$ implies $\Gamma \models^{\mathbf{UnivCFOL}}_{(S,F,F^c,P)} (\forall Y)\rho$. For the converse implication, assume that $\Gamma \not\models^{\mathbf{UnivFOL}}_{(S,F,P)} (\forall Y)\rho$. Let $\iota_Y : \Sigma \hookrightarrow \Sigma[Y]$ be the extension of $\Sigma = (S,F,P)$ with variables from $Y$. By Theorem 70, there exists a $\mathbb{S}^{\mathbf{FOL}}$-reachable model $M$ such that $M \models^{\mathbf{UnivFOL}}_{\Sigma[Y]} \Gamma$ and $M \not\models^{\mathbf{UnivFOL}}_{\Sigma[Y]} \rho$. Since $Y$ consists of variables of loose sorts, $M\upharpoonright_{\iota_Y} \in |\mathbb{M}od(S,F,F^{S^c},P)|$. Because $((S,F,F^c,P),E)$ is sufficient complete, $M\upharpoonright_{\iota_Y} \in |\mathbb{M}od(S,F,F^c,P)|$. We have $M\upharpoonright_{\iota_Y} \models^{\mathbf{UnivCFOL}}_{(S,F,F^c,P)} \Gamma$ and $M\upharpoonright_{\iota_Y} \not\models^{\mathbf{UnivCFOL}}_{(S,F,F^c,P)} (\forall Y)\rho$, which implies $\Gamma \not\models^{\mathbf{UnivCFOL}}_{(S,F,F^c,P)} (\forall Y)\rho$. □

The following borrowing result is from [21].

**Theorem 72** Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathbb{S}ig^\bullet \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S}en^\bullet : \mathbb{S}ig^\bullet \to \mathbb{S}et$ a sub-functor of $\mathbb{S}en : \mathbb{S}ig^\bullet \to \mathbb{S}et$ such that

(1) for all $\rho \in \mathbb{S}en(\Sigma)$ we have $\rho \models\!\mid \Gamma_\rho$ for some $\Gamma_\rho \subseteq \mathbb{S}en^\bullet(\Sigma)$.

Let $\mathbb{S}ig^{I^{sc}} \subseteq \mathbb{S}ig^{I^{pres}}$ be a full subcategory of presentation morphisms such that

(2) $(\Sigma,\Gamma) \in |\mathbb{S}ig^{I^{sc}}|$ whenever $(\Sigma,E) \in |\mathbb{S}ig^{I^{sc}}|$ and $E \subseteq \Gamma \subseteq \mathbb{S}en(\Sigma)$.

Let $(\phi,\alpha,\beta) : I \to I'$ be an institution morphism as in Definition 69 such that

(3) for all $\Sigma \in |\mathbb{S}ig|$, $\alpha_\Sigma$ is surjective, and

(4) for all $\Sigma \in |\mathbb{S}ig|$, $\Gamma' \subseteq \mathbb{S}en'(\phi(\Sigma))$ and $\rho' \in \mathbb{S}en'(\phi(\Sigma))$ if $(\Sigma, \alpha_\Sigma(\Gamma')) \in |\mathbb{S}ig^{I^{sc}}|$ and $\alpha_\Sigma(\rho') \in \mathbb{S}en^\bullet(\Sigma)$ then $\alpha_\Sigma(\Gamma') \models_\Sigma \alpha_\Sigma(\rho')$ iff $\Gamma' \models'_{\phi(\Sigma)} \rho'$.

Let $\mathcal{T} \subseteq \mathbb{S}ig^\bullet$ and $\mathcal{H} \subseteq \mathbb{S}ig$ be two broad subcategories of signature morphisms such that

(5) $\mathbb{S}ig$ has $(\mathcal{T},\mathcal{H})$-pushouts that are preserved by $\phi$,

(6) $\mathbb{S}ig^\bullet$ is closed to $(\mathcal{T},\mathcal{H})$-pushouts, i.e. for all pushouts of signature morphisms $\{\Sigma_2 \overset{\chi}{\leftarrow} \Sigma_0 \overset{\varphi}{\rightarrow} \Sigma_1, \Sigma_1 \overset{\chi_1}{\rightarrow} \Sigma \overset{\varphi_2}{\leftarrow} \Sigma_2\}$ such that $\varphi \in \mathcal{T} \subseteq \mathbb{S}ig^\bullet$ and $\chi \in \mathcal{H}$, we have $\varphi_2 \in \mathbb{S}ig^\bullet$,

(7) the inclusion functor $\mathbb{S}ig^{I^{sc}} \hookrightarrow \mathbb{S}ig^{I^{pres}}$ lifts $(\mathcal{T}^{pres},\mathcal{H}^{pres})$-pushouts.

Then the institution $I^{sc}$ has

(i) $(\mathcal{T}^{pres},\mathcal{H}^{pres})$-CRI whenever $I'$ has $(\phi(\mathcal{T}),\phi(\mathcal{H}))$-CRI, and

(ii) $(\mathcal{T}^{pres},\mathcal{H})$-CI whenever $I'$ has $(\phi(\mathcal{T}),\phi(\mathcal{H}))$-CI.

In concrete examples, $I$ is a constructor-based institution such as **UnivCFOL**. $(\phi,\alpha,\beta)$ is the forgetful institution morphism defined from the constructor-based institution to its base institution, such as $\Delta_{\mathbf{UnivCFOL}}$. $\mathbb{S}ig^\bullet$ is the broad subcategory of signature morphisms that encapsulate the constructors. $\mathbb{S}en^\bullet$ is the sentence sub-functor that maps each signature to the set of sentences free of quantification over variables of constrained sorts. $\mathbb{S}ig^{I^{sc}}$ is the full subcategory of sufficient complete presentations.

The following example shows that if we do not restrict $\mathbb{S}ig^\bullet$ to signature morphisms that encapsulate constructors then $\mathbb{S}en^\bullet : \mathbb{S}ig^\bullet \to \mathbb{S}et$ is not a functor, and our results may not hold.

**Example 73** Consider the following example of signature extension with a constructor:


In the above diagram, the abbreviations `op` and `constr` stand for operation and constructor, respectively. Let $\Sigma = dom(\iota)$, $\Sigma' = codom(\iota)$, and $\mathbb{S}en^\bullet$ the sub-functor that maps each signature to the set of sentences free of quantification over variables of constrained sorts. Note that $(\forall x)x = x \in \mathbb{S}en^\bullet(\Sigma)$ but $\iota((\forall x)x = x) \notin \mathbb{S}en^\bullet(\Sigma')$.

Condition (1) of Theorem 72 can be easily verified in our concrete examples.

**Lemma 74** [21] For all $(\forall X)\rho \in \mathbb{S}en^{\mathbf{UnivCFOL}}(\Sigma)$, where $\Sigma = (S, F, F^c, P)$ and $X$ is a finite set of variables of constrained sorts, we have $(\forall X)\rho \models\!\mid \Gamma_{(\forall X)\rho}$, where $\Gamma_{(\forall X)\rho}$ is defined as follows:

$$\Gamma_{(\forall X)\rho} = \{(\forall Y)\theta(\rho) \mid \theta : X \to T_{(S,F^c)}(Y),\ Y \text{ is a finite set of loose variables}\}$$

**Corollary 75** We have the following interpolation results:

(1) $\mathbf{UnivCFOL}^{sc}$ has $((iee*)^{pres}, (****)^{pres})$-CRI,

(2) $\mathbf{UnivCFOL}^{sc}$ has $((**e*)^{pres}, (iiei))$-CI.

PROOF. We set the parameters of Theorem 72. $I = \mathbf{UnivCFOL}$, $I' = \mathbf{UnivFOL}$, and $(\phi, \alpha, \beta) = \Delta_{\mathbf{UnivCFOL}}$. $\mathbb{S}ig^\bullet$ is the broad subcategory of signature morphisms consisting of $(**e*)$-morphisms. $\mathbb{S}en^\bullet$ is the sub-functor that maps each signature to the set of sentences free of quantification over variables of constrained sorts. $\mathbb{S}ig^{I^{sc}}$ is the full subcategory of sufficient complete presentations. Condition (2) of Theorem 72 holds because the sufficient complete property of a presentation is not be changed by adding new sentences. For all **UnivCFOL** signatures $(S, F, F^c, P)$, $\alpha_{(S,F,F^c,P)}$ is the identity, and in particular, a surjection. By Corollary 71, condition (4) of Theorem 72 is satisfied.

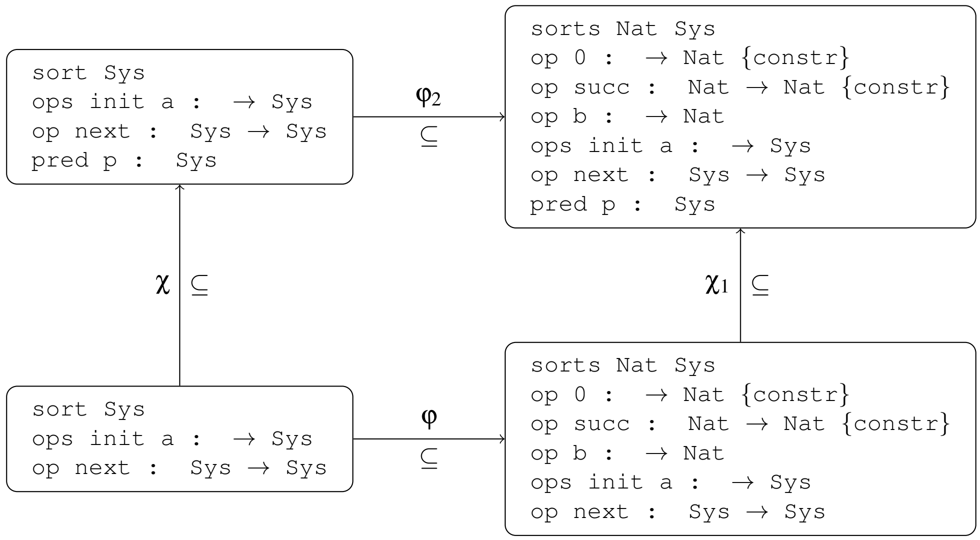
(1) By Proposition 59, **UnivCFOL** has $((iee*), (****))$-pushouts which are mapped to $((ie*), (***))$-pushouts by $\phi$. By Proposition 59 again, $(**e*)$-morphisms are closed to $((iee*), (****))$-pushouts. By Proposition 62, the inclusion functor $\mathbb{S}ig^{I^{sc}} \hookrightarrow \mathbb{S}ig^{I^{pres}}$ lifts $((iee*)^{pres}, (****)^{pres})$-pushouts. By Remark 68, **UnivFOL** has $((ie*), (***))$-CRI, and by Theorem 72, $\mathbf{UnivCFOL}^{sc}$ has $((iee*)^{pres}, (****)^{pres})$-CRI.

(2) By Proposition 60, **UnivCFOL** has $((**e*), (iiei))$-pushouts which are mapped to $((***), (iii))$-pushouts by $\phi$. By Proposition 60 again, $(**e*)$-morphisms are closed to $((**e*), (iiei))$-pushouts. By Proposition 63, the inclusion functor $\mathbb{S}ig^{I^{sc}} \hookrightarrow \mathbb{S}ig^{I^{pres}}$ lifts $((**e*)^{pres}, iiei)$-pushouts. By Remark 68, the institution **UnivFOL** has $((***), (iii))$-CI. By Theorem 72, $\mathbf{UnivCFOL}^{sc}$ has $((**e*)^{pres}, (iiei))$-CI.

□

The following example shows that without sufficient completeness assumption, an interpolant may not be found.

**Example 76** Consider the following pushout of **CFOL** of signature morphisms:




In the above diagram, the abbreviations `ops` and `pred` stand for operations and predicate, respectively. Let $\Sigma_0 = dom(\varphi) = dom(\chi)$, $\Sigma_1 = codom(\varphi)$, $\Sigma_2 = codom(\chi)$ and $\Sigma = codom(\chi_1) = codom(\varphi_2)$. Note that $\varphi : \Sigma_0 \to \Sigma_1$ is a $(iee*)$-morphism as no "new" (ordinary) operation and constructor symbols are introduced for "old" sorts. Define $E_1 \overset{def}{=} \{(\mathtt{b} = \mathtt{succ}^{\mathtt{n}}\ \mathtt{0}) \Rightarrow (\mathtt{a} = \mathtt{next}^{\mathtt{n}}\ \mathtt{init}) \mid \mathtt{n} \in \mathbb{N}\}$, $E_2 \overset{def}{=} \{\mathtt{p(a)}\}$ and $\Gamma_2 \overset{def}{=} \{\mathtt{p(next}^{\mathtt{n}}\ \mathtt{init)} \mid \mathtt{n} \in \mathbb{N}\}$.

The presentation $(\Sigma_1, \emptyset)$ is not sufficient complete because there are no equations to define the value of $b$. The presentation $(\Sigma_2, \emptyset)$ is sufficient complete because the signature $\Sigma_2$ has no constructors. Since all $\Sigma$-models $M$ have the carrier sets for the sort `Nat` consisting of interpretations of $\mathtt{s}^{\mathtt{n}}\mathtt{0}$, where $\mathtt{n} \in \mathbb{N}$, we have $M \models_\Sigma \mathtt{b} = \mathtt{s}^{\mathtt{m}}\mathtt{0}$ for some $\mathtt{m} \in \mathbb{N}$. If $M \models_\Sigma E_1$ then $M \models_\Sigma \mathtt{a} = \mathtt{next}^{\mathtt{m}}\mathtt{init}$ for some $\mathtt{m} \in \mathbb{N}$. If $M \models_\Sigma E_1 \cup \Gamma_2$ we get $M \models_\Sigma \mathtt{p(a)}$. Since $M$ was arbitrarily chosen, $E_1 \cup \Gamma_2 \models_\Sigma E_2$. In order to prove that the above pushout of signature morphisms is not a CRI square we recall the compactness property of an institution.

**Definition 77** An institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is *compact* when for every signature $\Sigma$, each set of sentences $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and any sentence $\rho \in \mathbb{S}en(\Sigma)$ if $\Gamma \models_\Sigma \rho$ then there exists a finite set $\Gamma_f \subseteq \Gamma$ such that $\Gamma_f \models_\Sigma \rho$.

**Remark 78** According to [10] and [12], **FOL** is compact.

Assume that there exists $E_0 \subseteq \mathbb{S}en(\Sigma_0)$ such that $E_1 \models^{\mathbf{UnivCFOL}}_{\Sigma_1} E_0$ and $E_0 \cup \Gamma_2 \models^{\mathbf{UnivCFOL}}_{\Sigma_2} E_2$. Notice that for all finite sets $\Gamma_f \subseteq \{\neg(\mathtt{a} = \mathtt{next}^{\mathtt{n}}\mathtt{init}) \mid n \in \mathbb{N}\}$ the set of sentences $E_1 \cup \Gamma_f$ is consistent in **UnivCFOL**. It follows that for all finite sets $\Gamma_f \subseteq \{\neg(\mathtt{a} = \mathtt{next}^{\mathtt{n}}\mathtt{init}) \mid n \in \mathbb{N}\}$ the set of sentences $E_0 \cup \Gamma_f$ is consistent in **UnivCFOL**. Since $\Sigma_0$ has no constructors, for all finite sets $\Gamma_f \subseteq \{\neg(\mathtt{a} = \mathtt{next}^{\mathtt{n}}\mathtt{init}) \mid n \in \mathbb{N}\}$ the set of sentences $E_0 \cup \Gamma_f$ is consistent in **FOL**. By compactness of **FOL**, $E_0 \cup \{\neg(\mathtt{a} = \mathtt{next}^{\mathtt{n}}\mathtt{init}) \mid n \in \mathbb{N}\}$ is consistent in **FOL**. Let $M \in |\mathbb{M}od^{\mathbf{FOL}}(\Sigma_0)| = |\mathbb{M}od^{\mathbf{UnivCFOL}}(\Sigma_0)|$ such that $M \models_{\Sigma_0} E_0 \cup \{\neg(\mathtt{a} = \mathtt{next}^{\mathtt{n}}\mathtt{init}) \mid n \in \mathbb{N}\}$. Let $N \in \mathbb{M}od^{\mathbf{UnivCFOL}}(\Sigma_2)$ such that

- $N_{\mathtt{Sys}} = M_{\mathtt{Sys}}$, $N_{\mathtt{init}} = M_{\mathtt{init}}$, $N_{\mathtt{a}} = M_{\mathtt{a}}$, $N_{\mathtt{next}} = M_{\mathtt{next}}$, and
- $N_{\mathtt{p}} = \{M_{\mathtt{next}^{\mathtt{n}}\mathtt{init}} \mid n \in \mathbb{N}\}$.

Since $N$ interprets all symbols in $\Sigma_0$ as $M$, $N \models_{\Sigma_2} E_0 \cup \{\neg(\mathtt{a} = \mathtt{next}^{\mathtt{n}}\mathtt{init}) \mid n \in \mathbb{N}\}$. By the definition of $N_{\mathtt{p}}$, $N \models_{\Sigma_2} \Gamma_2$. Since $N \models_{\Sigma_2} \{\neg(\mathtt{a} = \mathtt{next}^{\mathtt{n}}\mathtt{init}) \mid n \in \mathbb{N}\}$, by the definition of $N_{\mathtt{p}}$, $N \not\models_{\Sigma_2} \mathtt{p}(\mathtt{a})$. It follows that $E_0 \cup \Gamma_2 \not\models^{\mathbf{UnivCFOL}}_{\Sigma_2} E_2$. Hence, the above pushout of signature morphism is not a CRI square. If we allow disjunction of countable sets of sentences then the interpolant would be $\bigvee_{\mathtt{n}\in\mathbb{N}} (\mathtt{a} = \mathtt{next}^{\mathtt{n}}\mathtt{init})$.

# 6 Conclusions

We have lifted the DLSP from the conventional model theory to the institution-independent framework by developing an abstract method for proving DLST in a setting provided by institution theory. This method is applied to many-sorted first-order logic, preorder algebra, and higher-order logic with intensional Henkin semantics, but more applications are expected such as membership algebra [31], order-sorted algebra [19] and other combinations of these logics.

The method for proving DLSP within an arbitrary institution satisfying the conditions described in Theorem 3 is very general but it has some limitations. There are examples of more refined institutions for which we believe that the standard methods for proving DLSP cannot be replicated. In addition to the first method for proving DLSP we developed another one, by transporting the property (backwards) along an institution comorphism. The applicability power of this borrowing method is illustrated by deriving the DLSP for partial algebra and higher-order logic with Henkin semantics.

One major application of the DLST is interpolation in logics with constructors and universally quantified sentences of the form $(\forall X)\rho$, where $\rho$ is a quantifier-free formula. In [21] we have proved that interpolation holds in logics with constructors and Horn sentences, of the form $(\forall X) \wedge H \Rightarrow C$, where $H$ is a set of atomic formulas and $C$ is an atomic formula. The borrowing interpolation theorem of [21] is very general and it can be applied not only to Horn sentences but to all universal sentences. This is one of the important aspects of universal approach to logic (and implicitly of institution-independent approach to logic). We believe that these results can be naturally extended to institutions with sort generation constraints, such as the CASL institution.

The abstract results developed in [22] are applicable to institutions for which the cardinality of any signature is countable and the signature morphisms in $\mathcal{D}$ are conservative. In other words, the DLST theorem proved in [22] is not applicable to **FOL**, but to **FOL**$'$, the restriction of **FOL** to signatures consisting of a countable number of symbols and models with non-empty carriers. On the other hand, our general theorem can be used to prove DLSP for **FOL**$'$

Lindström theorem characterises first-order logic in terms of model-theoretic conditions such as Compactness and DLSP. This result says that any extension of first-order logic satisfies Compactness and the DLSP iff it is no more expressive than first-order logic. In [10] it is proved an institution-independent compactness result based on the ultraproduct construction on models. The present contribution shows that DLSP holds also in an arbitrary institution under appropriate conditions. First-order logic, in its many-sorted form, is an instance of both results of [10] and Theorem 46, which suggests that the classical proof of Lindström theorem can be adapted to many-sorted first-order logic. One of the future directions of research concerns an institution-independent version of Lindström theorem. Also, we are planning to investigate interpolation in first-order logics with constructors such as **CFOL**.

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