Regularization of $\mathbf{P}_0$-functions in Box Variational Inequality Problems

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Abstract

In two recent papers, Facchinei [7] and Facchinei and Kanzow [8] have shown that for a continuously differentiable P_0-function f, the nonlinear complementarity problem NCP(f_ε) corresponding to the regularization $f_\varepsilon(x) := f(x) + \varepsilon x$ has a unique solution for every $\varepsilon > 0$, that dist $(x(\varepsilon), \text{SOL}(f)) \to 0$ as $\varepsilon \to 0$ when the solution set SOL(f) of NCP(f) is nonempty and bounded, and NCP(f) is stable if and only if the solution set is nonempty and bounded. They prove these results via the the Fischer function and the Mountain Pass Theorem. In this paper, we generalize these NCP results to a Box Variational Inequality Problem corresponding to a continuous $\mathbf{P}_0$-function where the regularization is described by an integral. We also describe an upper semicontinuity property of the inverse of a weakly univalent function and study its consequences.

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1 Introduction

Consider a continuous function $f : R^n \to R^n$ and a rectangular box K in R^n. Then the *Box Variational Inequality Problem*, denoted by BVI(f, K), is to find an $x^* \in K$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0 \quad \text{for all} \quad x \in K. \tag{1}$$

When $K = R^n_+$, this problem reduces to the nonlinear complementarity problem NCP(f): Find $x^* \in R^n$ such that

$$x^* \geq 0,\; f(x^*) \geq 0, \quad \text{and} \quad \langle f(x^*), x^* \rangle = 0. \tag{2}$$

Both the NCP and BVI have been extensively studied in the literature, see [4], [5], [9], [10], [11], [13], [18], [19], [21], and the references therein.

We say that f is a $\mathbf{P}_0$ ($\mathbf{P}$)-function if for every pair (x, y) with $x \neq y$,

$$\max_{x_i \neq y_i} (x - y)_i [f_i(x) - f_i(y)] \geq 0 \quad (> 0). \tag{3}$$

Generalizing earlier results for monotone functions, Facchinei [7] and Facchinei and Kanzow [8] have shown the following in the NCP setting: Consider a continuously differentiable $\mathbf{P}_0$-function f and its Tikhonov regularization $f_\varepsilon(x) := f(x) + \varepsilon x$. Then

(a) NCP(f_ε) has a unique solution $x(\varepsilon)$ for each $\varepsilon > 0$;

(b) When the solution set SOL(f) of NCP(f) is nonempty and bounded,

$$\text{dist } (x(\varepsilon), \text{SOL}(f)) \to 0 \quad \text{as} \quad \varepsilon \to 0;$$

(c) SOL(f) is stable if and only if it is nonempty and bounded.

Although item (a) follows from a result of Megiddo and Kojima (Thm. 3.4, [17]), the way of proving these results in the cited papers via the the Fischer function and the Mountain Pass Theorem is quite interesting and novel. In a related paper, D. Sun [21] carries out an algorithmic analysis of a continuously differentiable $\mathbf{P}_0$ complementarity problem via regularization techniques.

In this paper, we generalize the results of Facchinei and Kanzow in several ways. We consider a BVI instead of a NCP, assume only continuous $\mathbf{P}_0$-property of f, and deal with integral regularizations of the fixed point map of BVI (1) of the form

$$\widehat{F}_\varepsilon(x) := \int_R \{x - \Pi_K(x - f(x) - \varepsilon x - \varepsilon s\, e)\} d\mu(s)$$

where e is the vector of ones in R^n, and μ is a Borel measure on R. Our analysis is based on degree theory and the classical result that a coercive local homeomorphism of R^n is a global homeomorphism of R^n. In contrast to our theoretical analysis, H.-D. Qi [18] makes an algorithmic study of a BVI with a continuously differentiable $\mathbf{P}_0$-function via the mountain pass theorem and the normal map.

2 Preliminary results

2.1 P and P_0-properties

Throughout this paper, K denotes a rectangular box in R^n, i.e.,

$$K = K_1 \times K_2 \cdots \times K_n$$

where each K_i is a closed interval in R. It is well known that BVI(f, K) is equivalent to finding a zero of the (fixed point map) $\widehat{f}$ defined by

$$\widehat{f}(x) := x - \Pi_K(x - f(x)) \tag{4}$$

where Π_K denotes the (orthogonal) projection onto K. We note that when $K = R^n_+$ (the non-negative orthant),

$$\widehat{f}(x) = x \wedge f(x)$$

where '$\wedge$' denotes the componentwise minimum of vectors involved.

Given a continuous function $f : R^n \to R^n$, $\varepsilon > 0$ and a (positive) Borel measure μ on R [20] with

$$\mu(R) = 1 \text{ and } \Delta := \textstyle\int_R |s| d\mu(s) < \infty,$$

we define the following weak and strong regularizations of $\widehat{f}$:

$$\widehat{f}_\varepsilon(x) := \int_R \{x - \Pi_K(x - f(x) - \varepsilon s e)\} d\mu(s) \tag{5}$$

and

$$\widehat{F}_\varepsilon(x) := \int_R \{x - \Pi_K(x - f(x) - \varepsilon x - \varepsilon s e)\} d\mu(s) \tag{6}$$

where the integration is performed componentwise. Such 'transforms' appear in smoothing methods for NCP and BVI, see [4], [5], [11] and the references therein. In view of the following inequalities, we see that $\widehat{f}_\varepsilon$ is a uniform approximation of $\widehat{f}$ on R^n, and that $\widehat{F}_\varepsilon$ is a uniform approximation of $\widehat{f}$ on bounded subsets of R^n:

$$||\widehat{f}_\varepsilon(x) - \widehat{f}(x)|| \leq \varepsilon\sqrt{n}\Delta \quad (x \in R^n)$$

and for any compact set E, there is a constant C such that

$$||\widehat{F}_\varepsilon(x) - \widehat{f}(x)|| \leq \varepsilon C \quad (x \in E). \tag{7}$$

We also note that when $K = R^n_+$ and μ is the point mass at the origin (so that $\int g \, d\mu = g(0)$ for every continuous g on R),

$$\widehat{f}_\varepsilon(x) = x \wedge f(x) \quad \text{and} \quad \widehat{F}_\varepsilon(x) = x \wedge [f(x) + \varepsilon x].$$

Other special cases are obtained by putting $d\mu = \rho(s)ds$ where ρ is a density function on R (see [4], [11]).

In this section, we establish the $\mathbf{P}$ and $\mathbf{P}_0$-properties of $\widehat{f}_\varepsilon$ and $\widehat{F}_\varepsilon$ given by (5) and (6).

Proposition 1 *For a continuous function* $f : R^n \to R^n$*, let*

$$\theta(x) := x - \Pi_K(x - f(x)),$$
$$\theta(x, \varepsilon, s) := x - \Pi_K(x - f(x) - \varepsilon se) \quad (s \in R,\ \varepsilon > 0), \text{ and}$$
$$\phi(x, \varepsilon, s) := x - \Pi_K(x - f(x) - \varepsilon x - \varepsilon se) \quad (s \in R,\ \varepsilon > 0).$$

Then

(a) $\theta(x)$ *and* $\theta(x, \epsilon, s)$ *are* $\mathbf{P}$*-functions (*$\mathbf{P}_0$*-functions) in* x *(for fixed* s *and* ε*) whenever* f *is a* $\mathbf{P}$*-function (*$\mathbf{P}_0$*-function),*

(b) $\phi(x, \varepsilon, s)$ *is a* $\mathbf{P}$*-function in* x *(for fixed* s *and* ε*) when* f *is a* $\mathbf{P}_0$*-function.*

Proof. (a) Assume that f is a $\mathbf{P}$-function. Let $x \neq y$ and pick an i such that

$$(x_i - y_i)[f_i(x) - f_i(y)] > 0.$$

Without loss of generality, let $x_i > y_i$ so that $f_i(x) > f_i(y)$. We show that θ is a $\mathbf{P}$-function by showing that $(x_i - y_i)[\theta_i(x) - \theta_i(y)] > 0$. If the inequality were not true, then $\theta_i(x) \leq \theta_i(y)$ which means that

$$x_i - \Pi_{K_i}(x_i - f_i(x)) \ \leq\ y_i - \Pi_{K_i}(y_i - f_i(y)).$$

By considering all possible values of the quantities involved in the above expression, we can check (see for example, Appendix 1) that the above inequality cannot hold. We conclude that θ is a $\mathbf{P}$-function. A similar argument shows that θ is a $\mathbf{P}_0$-function when f is a $\mathbf{P}_0$-function. Since f is a $\mathbf{P}$ ($\mathbf{P}_0$)-function if and only if $f(x) + \varepsilon es$ is a $\mathbf{P}$ ($\mathbf{P}_0$)-function, we get the stated assertion about $\theta(x, \varepsilon, s)$.

(b) follows easily from (a) since $f(x) + \varepsilon x$ is a $\mathbf{P}$-function when f is a $\mathbf{P}_0$-function. □

Remark 1 The proof of the above proposition actually shows the following: Suppose f is a $\mathbf{P}$-function ($\mathbf{P}_0$-function) and $x \neq y$. If $x_i \neq y_i$ and

$$(x_i - y_i)[f_i(x) - f_i(y)] > 0 \ (\geq 0),$$

then for the same index i,

$$(x_i - y_i)[\theta_i(x, \varepsilon, s) - \theta_i(y, \varepsilon, s)] > 0 \ (\geq 0) \quad \text{for all } s \in R,\ \varepsilon > 0$$

and

$$(x_i - y_i)[\phi_i(x, \varepsilon, s) - \phi_i(y, \varepsilon, s)] > 0 \quad \text{for all } s \in R,\ \varepsilon > 0.$$

Proposition 2 *Given f and μ, let $\widehat{f}_\varepsilon$ and $\widehat{F}_\varepsilon$ be as defined in (5) and (6). Then the following statements hold:*

(a) If f is a $\mathbf{P}_0$($\mathbf{P}$)-function, then $\widehat{f}_\varepsilon$ is a $\mathbf{P}_0$ ($\mathbf{P}$)-function.

(b) If f is a $\mathbf{P}_0$-function, then $\widehat{F}_\varepsilon$ is a $\mathbf{P}$-function and hence univalent.

Proof. Let f be a $\mathbf{P}_0(\mathbf{P})$-function. Fix $x \neq y$ in R^n. Then there exists an index i such that $x_i \neq y_i$ and $(x_i - y_i)[f_i(x) - f_i(y)] \geq 0$ (> 0). From Remark 1, we see that

$$(x_i - y_i)[\theta_i(x, \varepsilon, s) - \theta_i(y, \varepsilon, s)] \geq 0 \ (> 0) \quad \text{for all } s \in R.$$

Since $\mu(R) > 0$, integration leads to

$$(x_i - y_i)[\widehat{f}_\varepsilon(x) - \widehat{f}_\varepsilon(y)]_i \geq 0 \ (> 0).$$

Thus we have (a). Item (b) is proved by applying (a) to the $\mathbf{P}$-function $f(x) + \varepsilon x$. The univalence (i.e., one-to-one) assertion follows from the $\mathbf{P}$-property. □

In the result below, we identify a condition under which $\widehat{f}_\varepsilon$ is a $\mathbf{P}$-function.

Proposition 3 *Suppose f is a $\mathbf{P}_0$-function and for each i, the closed interval K_i is either bounded below or above, and μ does not vanish on any infinite interval. Then $\widehat{f}_\varepsilon$ is a $\mathbf{P}$-function.*

Proof. Let $x \neq y$. From Remark 1 and the previous proposition we know that for some index i, $x_i \neq y_i$,

$$(x_i - y_i)[\theta_i(x, \varepsilon, s) - \theta_i(y, \varepsilon, s)] \geq 0 \ (> 0) \quad \text{for all } s \in R$$

and

$$(x_i - y_i)[\widehat{f}_\varepsilon(x) - \widehat{f}_\varepsilon(y)]_i \geq 0.$$

We claim that the latter inequality is strict. Assume the contrary and let without loss of generality, $x_i > y_i$ and $[\widehat{f}_\varepsilon(x) - \widehat{f}_\varepsilon(y)]_i = 0$. It follows that

$$\theta_i(x, \varepsilon, s) = \theta_i(y, \varepsilon, s) \quad a.e.\ \mu, \tag{8}$$

that is, the set of s where the above equality fails to hold has μ measure zero. Assume that K_i is bounded below by $l_i > -\infty$ (the case of K_i being bounded above is similar). Then for all s in some interval $[\delta, \infty)$, $x_i - f_i(x) - \varepsilon s \leq l_i$ and $y_i - f_i(y) - \varepsilon s \leq l_i$; hence for all such s, $\theta_i(x, \varepsilon, s) = x_i - l_i$ and $\theta_i(y, \varepsilon, s) = y_i - l_i$. Since $x_i - l_i \neq y_i - l_i$, we see from (8) that $\mu[\delta, \infty) = 0$ contradicting the assumption on μ. □

Corollary 1 *Suppose f is a $\mathbf{P}_0$-function and $K = R^n_+$. Then*

$$\widehat{f}(x) = x \wedge f(x)$$

and

$$\widehat{f}_\varepsilon(x) = \int_R x \wedge (f(x) + \varepsilon es) d\mu(s)$$

are $\mathbf{P}_0$-functions. Moreover, $\widehat{f}_\varepsilon$ will be a $\mathbf{P}$-function when one of the following conditions is satisfied:

(i) f is a $\mathbf{P}$-function;

(ii) μ does not vanish on any infinite interval of the form $[\delta, \infty)$.

Proof. The stated property of $\widehat{f}$ follows from Proposition 2(a) by taking μ to be the point mass at the origin. For the stated properties of $\widehat{f}_\varepsilon$ see Proposition 2(a) and the proof of Proposition 3. $\square$

Remark 2 The above Corollary says that the composition of a $\mathbf{P}_0$ ($\mathbf{P}$)-function and the min function is a $\mathbf{P}_0$ ($\mathbf{P}$)-function. Similar things can be said about the Fischer function. Recall that the ith component of the Fischer function $\psi : R^n \times R^n \to R^n$ is given by

$$\psi_i(a, b) = (a_i + b_i) - \sqrt{a_i^2 + b_i^2}.$$

We claim that *if f is a $\mathbf{P}$-function, then so is $\Psi(x) := \psi(x, f(x))$.*
To see this, let $x \neq y$. Then there exists an index i such that $(x_i - y_i)(f_i(x) - f_i(y)) > 0$. We show that for the same index i, $(x_i - y_i)(\Psi_i(x) - \Psi_i(y)) > 0$. Without loss of generality, we assume that $x_i > y_i$ and show that $\Psi_i(x) - \Psi_i(y) > 0$. Assume the contrary, $\Psi_i(x) \leq \Psi_i(y)$ and let, for simplicity, $\alpha = x_i$, $\beta = y_i$, $\gamma = f_i(x)$, $\delta = f_i(y)$. We have $\alpha > \beta$ and $\gamma > \delta$. Now $\Psi_i(x) \leq \Psi_i(y)$ leads to $(\alpha - \beta) + (\gamma - \delta) \leq \sqrt{\alpha^2 + \gamma^2} - \sqrt{\beta^2 + \delta^2}$. Squaring, simplifying, and using the inequality $(\alpha - \beta)(\gamma - \delta) > 0$, we get

$$\sqrt{\alpha^2 + \gamma^2}\sqrt{\beta^2 + \delta^2} < \alpha\beta + \gamma\delta$$

which upon squaring, leads to $(\alpha\delta - \beta\gamma)^2 < 0$. We conclude that Ψ is a $\mathbf{P}$-function. The $\mathbf{P}_0$-property is similarly established. Note that the zeros of Ψ are precisely the solutions of NCP(f).

Remark 3 Let f be a $\mathbf{P}_0$-function. Let $\eta(x) : R^n \to R^n$ be a function whose ith component function η_i is a function of x_i only and that it is strictly increasing in this variable. Then it is easily verified that for any $\varepsilon > 0$, $f(x) + \varepsilon\eta(x)$ is a $\mathbf{P}$-function. In particular $f(x) + \varepsilon\eta(x)$ is a $\mathbf{P}$-function where η is given, for some (disjoint) index sets I, J, and an $\overline{x} \in R^n$, by

$$\eta_i(x) = \begin{cases} -e^{-x_i} & \text{if } i \in I \\ e^{x_i} & \text{if } i \in J \\ (x_i - \overline{x}_i) & \text{if } i \notin I \cup J \end{cases}$$

2.2 An upper semicontinuity property

In Corollary 3 of this section, we describe an upper semicontinuity property of the (multivalued) inverse of the map $\widehat{f}$ corresponding to a $\mathbf{P}_0$-function f. We will need this result in the proof of the main theorem in Section 3. We deduce this Corollary 3 as a consequence of the following

Theorem 1 *Let $g : R^n \to R^n$ be weakly univalent, that is, g is continuous, and there exist one-to-one continuous functions $g_k : R^n \to R^n$ such that $g_k \to g$ uniformly on every bounded subset of R^n. Suppose that $q^* \in R^n$ such that*

$$g^{-1}(q^*) \text{ is nonempty and compact.}$$

Then for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any weakly univalent function h and for any q with

$$\sup_{\overline{\Omega}} ||h(x) - g(x)|| < \delta, \quad ||q - q^*|| < \delta \tag{9}$$

we have

$$\emptyset \neq h^{-1}(q) \subseteq g^{-1}(q^*) + \varepsilon \mathcal{B} \tag{10}$$

where $\mathcal{B}$ denotes the open unit ball in R^n and $\Omega := g^{-1}(q^*) + \varepsilon\mathcal{B}$. In particular, $h^{-1}(q)$ and $g^{-1}(q)$ are nonempty, connected, and uniformly bounded for q in a neighborhood of q^*.

Proof. Let $\varepsilon > 0$ and $\Omega := g^{-1}(q^*) + \varepsilon\mathcal{B}$. Under the stated assumptions on g, it follows from Remark (2) in Section 2 of [12] (together with the excision property of the degree) that

$$deg\,(g, \Omega, q^*) = \pm 1. \tag{11}$$

Let $\delta := \frac{1}{2}$ dist $(q^*, g(\partial\Omega))$ where $\partial\Omega$ denotes the boundary of Ω. Then for h and q satisfying (9),

$$\sup_{\overline{\Omega}} ||(h(x) - q) - (g(x) - q^*)|| < \text{ dist}\,(q^*, g(\partial\Omega))$$

and hence by the nearness property of the degree (Thm. 2.1.2, [15]),

$$deg\,(h, \Omega, q) = \pm 1.$$

It follows that $h(x) = q$ will have a solution in Ω and no solutions on $\partial\Omega$. We now claim that $h^{-1}(q) \subseteq \Omega$. The decomposition

$$h^{-1}(q) = [h^{-1}(q) \cap \Omega] \cup [h^{-1}(q) \cap (\overline{\Omega})^c]$$

shows that $h^{-1}(q) \cap \Omega$ is a nonempty, bounded, closed and open subset of $h^{-1}(q)$. It follows from Theorem 2 in [12] that $h^{-1}(q) = h^{-1}(q) \cap \Omega$, proving (10). The same theorem proves the connectedness of $h^{-1}(q)$. Finally putting $h = g$, we get the stated assertion about g. This completes the proof. □

The above theorem has a number of important consequences.

Corollary 2 *Let g be weakly univalent, Y be a closed convex subset of R^n such that $g^{-1}(Y)$ is bounded and*

$$\text{for some } q^* \in Y,\ g^{-1}(q^*) \neq \emptyset.$$

Then for each $q \in Y$, $g^{-1}(q) \neq \emptyset$ and $g^{-1}(Y)$ is connected.

Proof. For any fixed $q \in Y$, we consider the homotopy $H(x,t) := g(x) - [tq^* + (1-t)q]$. By (11) and the homotopy invariance of the degree, we conclude that the degree of $g(x) - q$ over a suitable bounded open set is nonzero proving the nonemptyness of $g^{-1}(q)$. It follows from Theorem 1 that g^{-1} is upper semicontinuous at each point of Y. Since Y is connected, $g^{-1}(Y)$ is closed, $g^{-1}(q)$ is connected for all $q \in Y$ and g^{-1} is upper semicontinuous at each point of Y, it follows that $g^{-1}(Y)$ is also connected. □

Remark 4 It follows from the above corollary that if a weakly univalent function $g : R^n \to R^n$ is proper (that is, inverse image of any compact set is compact), then it is onto.

To see our next consequence, consider a $\mathbf{P}_0$-function f. Then by Proposition 2(b), $\widehat{F}_\varepsilon$ is a univalent function. Letting $\varepsilon \to 0$ in (7), we see that $\widehat{f}$ is a weakly univalent function. Theorem 1 now gives the following

Corollary 3 *Let f be a continuous $\mathbf{P}_0$-function and let $\widehat{f}$ be given by (4). Suppose that $(\widehat{f}\,)^{-1}(q^*)$ is nonempty and compact. Then for each $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\emptyset \neq (\widehat{f}\,)^{-1}(q) \subseteq (\widehat{f}\,)^{-1}(q^*) + \varepsilon\mathcal{B} \tag{12}$$

for all q with $||q - q^*|| < \delta$. In particular, $(\widehat{f}\,)^{-1}(q)$ is nonempty and (uniformly) bounded for all q in a neighborhood of q^*.

Remark 5 Let f be as in the above corollary. Suppose that $(\widehat{f}\,)^{-1}(0)$ is nonempty and bounded. Then $(\widehat{f})^{-1}(q)$ is uniformly bounded for $||q||$ small. This can be described equivalently by means of level sets: For all small positive numbers α, the level sets

$$\{x : ||\widehat{f}(x)|| \leq \alpha\}$$

are bounded. Such a boundedness result has been used to analyze convergence in various iterative schemes, see e.g., [5] and [22].

Remark 6 As yet another illustration of Theorem 1, suppose f is a $\mathbf{P}_0$-function, and consider $\Psi(x) = \psi(x, f(x))$ where ψ is the Fischer function mentioned in Remark 2. We know from Remark 2 that Ψ is a $\mathbf{P}_0$-function and $(\Psi)^{-1}(0)$ is the solution set of the nonlinear complementarity problem NCP(f). By applying Theorem 1, one can state a result similar to Corollary 3 with Ψ

in place of $\widehat{f}$. A modification of Remark 5 for Ψ gives the following result: For all small positive numbers α, the level sets

$$\{x : ||\Psi(x)|| \leq \alpha\}$$

are bounded. This result generalizes Lemma 4.3 in [7] proved for a continuously differentiable $\mathbf{P}_0$-function via the Mountain Pass Theorem.

We now state an upper semicontinuity property of the solution set of a BVI.

Corollary 4 *Let f be a continuous $\mathbf{P}_0$-function and $q \in R^n$. Let* $\mathrm{BVI}(f, K, q)$ *and* $\mathrm{SOL}(f, K, q)$ *denote, respectively, the box variational inequality problem corresponding to the function $f(x) + q$ on K and its solution set. Suppose that for $q^* \in R^n$,*

$$\mathrm{SOL}(f, K, q^*) \text{ is nonempty and bounded.}$$

Then for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\emptyset \neq \mathrm{SOL}(f, K, q) \subseteq \mathrm{SOL}(f, K, q^*) + \varepsilon\mathcal{B} \tag{13}$$

for all q with $||q - q^*|| < \delta$. In particular, $\mathrm{SOL}(f, K, q)$ *is nonempty, connected, and (uniformly) bounded for all q in a neighborhood of q^*.*

Proof. The result follows from Theorem 1 by putting

$$g(x) = x - \Pi_K(x - f(x) - q^*), \quad \text{and} \quad h(x) = x - \Pi_K(x - f(x) - q).$$

□

When the above corollary is specialized to $K = R^n_+$, we get the following

Corollary 5 *If the nonlinear complementarity problem* $\mathrm{NCP}(f)$ *corresponding to a continuous $\mathbf{P}_0$-function has nonempty bounded solution set, then the problem is strictly feasible, that is, there exists an x^* such that*

$$x^* > 0 \quad \textit{and} \quad f(x^*) > 0.$$

Proof. In the previous corollary, we put $K = R^n_+$, $q^* = 0$, and take $q < 0$ sufficiently close to zero. Then $\mathrm{SOL}(f, R^n_+, q)$ is nonempty and every solution u in $\mathrm{SOL}(f, R^n_+, q)$ satisfies $u \geq 0$ and $f(u) \geq -q > 0$. By continuity we produce an x^* satisfying the above listed properties. □

Remark 7 In the above corollary we considered an NCP. The BVI version reads as follows. Suppose K is a rectangular box with 0^+K denoting the recession cone of K and $(0^+K)^*$ denoting the dual of 0^+K. If $(0^+K)^*$ has nonempty interior and f is a continuous $\mathbf{P}_0$-function with $\mathrm{SOL}(f, K, 0)$ nonempty and bounded, then there exists an x^* such that

$$x^* \in K \quad \text{and} \quad f(x^*) \in \operatorname{int}(0^+K)^*.$$

This can be seen by taking $q \in -\mathrm{int}\,(0^+K)^*$ that is close to zero and applying (13) to get a solution x^* of $\mathrm{SOL}(f, K, q)$. The inequality (1) shows that $f(x^*) + q \in (0^+K)^*$ from which we get the stated properties of x^*.

Now consider a continuous monotone f so that for all x, y,

$$\langle f(x) - f(y), x - y \rangle \geq 0.$$

In this setting, it is well known that $\mathrm{NCP}(f)$ is solvable with a bounded solution set whenever it has a strictly feasible solution (see Theorem 4.1, [14] or Theorem 3.4, [13]). The above corollary proves the converse of this result. While we have used degree theoretic considerations to prove this converse, McLinden has proved this converse in the setting of maximal monotone operators (Theorem 4, [16]) and Chen, Chen, and Kanzow [3] prove it (for a continuously differentiable f) using the Fischer function and the Mountain Pass Theorem. Easy examples of affine $\mathbf{P}_0$-functions show that the converse statement in the above corollary need not hold.

2.3 A coercivity property

We saw in Proposition 2 that for a continuous $\mathbf{P}_0$-function f, $\widehat{F}_\varepsilon$ is univalent. Towards establishing the homeomorphism property of $\widehat{F}_\varepsilon$, we prove the following result.

Proposition 4 *Suppose f is a $\mathbf{P}_0$-function and define $\widehat{F}_\varepsilon$ as in (6). Then $\widehat{F}_\varepsilon$ is coercive on R^n, that is, for any sequence $\{x^k\}$ with $||x^k|| \to \infty$, we have $||\widehat{F}_\varepsilon(x^k)|| \to \infty$.*

Proof. Fix $\varepsilon > 0$. To show that $\widehat{F}_\varepsilon$ is coercive, we follow an argument given in the proof of Proposition 3.4 [7]. Let $\{x^k\}$ be any sequence such that $||x^k|| \to \infty$. Passing through a subsequence, if necessary, we may suppose that there exists an index set J such that for each $i \in J$, $|x_i^k| \to \infty$ as $k \to \infty$ and for $i \notin J$, $\{x_i^k\}$ is bounded. Define a bounded sequence $\{y^k\}$ as follows:

$$y_i^k := \begin{cases} 0 & \text{if } i \in J \\ x_i^k & \text{if } i \notin J \end{cases}$$

Since $x^k \neq y^k$ for all large k, we can use the $\mathbf{P}_0$-property of f to get an index $i \in J$ so that without loss of generality,

$$x_i^k[f_i(x^k) - f_i(y^k)] = (x_i^k - y_i^k)[f_i(x^k) - f_i(y^k)] \geq 0$$

for all k. For simplicity we may take $i = 1$ and note that $|x_1^k| \to \infty$. We assume without loss of generality, x_1^k converges either to ∞ or to $-\infty$.

Suppose that x_1^k goes to ∞. Then the above inequality shows (assuming $x_1^k > 0$ for all k) that $f_1(x^k)$ is bounded below by $\alpha := \inf_k f_1(y^k)$. Now consider

$$(\widehat{F}_\varepsilon)_1(x^k) = \int_R \{x_1^k - \Pi_{K_1}[x_1^k - f_1(x^k) - \varepsilon x_1^k - \varepsilon s]\} d\mu(s). \tag{14}$$

This integral can be written as the sum of integrals over A_k, B_k, and C_k where

$$A_k := \{s : x_1^k - f_1(x^k) - \varepsilon x_1^k - \varepsilon s < l_1\}$$

when K_1 is bounded below with $l_1 = \inf K_1$,

$$B_k := \{s : x_1^k - f_1(x^k) - \varepsilon x_1^k - \varepsilon s \in K_1\}$$

and

$$C_k := \{s : x_1^k - f_1(x^k) - \varepsilon x_1^k - \varepsilon s > u_1\}$$

when K_1 is bounded above with $u_1 = \sup K_1$.

Note that some of these sets may be empty. When K_1 is bounded, in view of $\mu(R) = 1$, the integral (14) behaves like x_1^k and hence goes to infinity as $k \to \infty$. When $K_1 = R$, the integral reduces to $f_1(x^k) + \varepsilon x_1^k + \varepsilon C$ where C is a constant. Since $f_1(x^k)$ is bounded below, even in this case also, the integral goes to infinity. Now consider the case when K_1 is bounded below but not above. Then C_k is empty, the integral in (14) reduces to

$$\int_{A_k} (x_1^k - l_1) d\mu(s) + \int_{B_k} [f_1(x^k) + \varepsilon x_1^k + \varepsilon s] d\mu(s) = x_1^k[\mu(A_k) + \varepsilon\mu(B_k)] + \gamma_k$$

where

$$\gamma_k = -l_1\mu(A_k) + f_1(x^k)\mu(B_k) + \varepsilon \int_{B_k} s d\mu(s) \geq \alpha\mu(B_k) - l_1 - \varepsilon\Delta$$

where we recall that $\alpha \leq f_1(x^k)$, $\Delta := \int_R |s| d\mu(s)$, and $\mu(R) = 1$. Since $\mu(A_k) + \mu(B_k) = 1$, there exists a positive number δ such that $\mu(A_k) + \varepsilon\mu(B_k) \geq \delta$ for all k large. Hence the integral in (14) exceeds $x_1^k\delta +$ constant. It follows that the integral goes to infinity as $k \to \infty$. Similar arguments can be used when K_1 is unbounded below but bounded above. Thus we have shown that as $x_1^k \to \infty$, $(\widehat{F}_\varepsilon)_1(x^k) \to \infty$. The proof that $(\widehat{F}_\varepsilon)_1(x^k) \to -\infty$ as $x_1^k \to -\infty$ is similar; we omit the details. Thus we have shown that for any sequence $\{x^k\}$ going to infinity in the norm, $\{\widehat{F}_\varepsilon(x^k)\}$ goes to infinity in the norm through a subsequence. This proves that for any such sequence $\{x^k\}$, $||\widehat{F}_\varepsilon(x^k)|| \to \infty$ as $k \to \infty$. This completes the proof. □

3 The Main Result

We now consider BVI(f, K) and denote its solution set by SOL(f, K).

For our main theorem below, we introduce the following stability concepts. The first one appears in [7].

Definition 1 *We say that* $\mathrm{BVI}(f, K)$ *is linearly stable if for every* $\varepsilon > 0$*, there exists a* $\delta^* > 0$ *such that for any continuous function* g *with*

$$||g(x) - f(x)|| \leq \delta^*(1 + ||x||) \quad \forall\, x \in \mathrm{SOL}(f, K) + \varepsilon\mathcal{B},$$

$\mathrm{BVI}(g, K)$ *has a solution in* $\mathrm{SOL}(f, K) + \varepsilon\mathcal{B}$.

Definition 2 *We say that* $\mathrm{BVI}(f, K)$ *is directionally stable if for every* $\varepsilon > 0$*, and every continuous function* h*, there exists a* $\overline{\delta} > 0$ *such that for* $0 \leq \delta \leq \overline{\delta}$*,* $\mathrm{BVI}(f + \delta h, K)$ *has a solution in* $\mathrm{SOL}(f, K) + \varepsilon\mathcal{B}$.

We are now ready for our main result.

Theorem 2 *Let* $f : R^n \to R^n$ *be a continuous* $\mathbf{P}_0$*-function and let* $\widehat{F}_\varepsilon$ *be as in (6). Then the following statements hold:*

(a) For each $\varepsilon > 0$*, the equation* $\widehat{F}_\varepsilon(x) = 0$ *has a unique solution* $x(\varepsilon)$*. Moreover, the mapping* $\varepsilon \mapsto x(\varepsilon)$ *from* $(0, \infty)$ *to* R^n *is continuous.*

(b) If $\mathrm{SOL}(f, K)$ *is nonempty and bounded, then dist* $(x(\varepsilon), \mathrm{SOL}(f, K)) \to 0$ *as* $\varepsilon \to 0$.

(c) $(i) \Longleftrightarrow (ii) \Longrightarrow (iii)$ *where*

(i) $\mathrm{BVI}(f, K)$ *is directionally stable.*

(ii) $\mathrm{SOL}(f, K)$ *is nonempty and bounded.*

(iii) $\mathrm{BVI}(f, K)$ *is linearly stable.*

Moreover, when K_i *is bounded below for each* i*,* $(iii) \Longrightarrow (ii)$.

Proof. (a) For $\varepsilon > 0$, Propositions 2 and 4 show that $\widehat{F}_\varepsilon$ is univalent and coercive. Since coercivity in R^n is the same as properness (that is, inverse image of any compact set is compact), by a classical result of Banach and Mazur ([1] or Theorem 5.1.4, [2]), it follows that $\widehat{F}_\varepsilon$ is a homeomorphism of R^n and hence $\widehat{F}_\varepsilon(x) = 0$ will have a unique solution. An application of Theorem 1 with $q = q^* = 0$, $g := \widehat{F}_{\varepsilon^*}$, and $h := \widehat{F}_\varepsilon$ proves the continuity of $x(\varepsilon)$ at any $\varepsilon^* > 0$.
(b) Now let $\mathrm{SOL}(f, K)$ be nonempty and bounded. For any $\zeta > 0$, let $D = \mathrm{SOL}(f, K) + \zeta\mathcal{B}$. Then for all sufficiently small ε, $\widehat{F}_\varepsilon$ is close to $\widehat{f}$ on $\overline{D}$, and hence by Theorem 1, $(\widehat{F}_\varepsilon)^{-1}(0) \subseteq (\widehat{f})^{-1}(0) + \zeta\mathcal{B}$. Since $x(\varepsilon) = (\widehat{F}_\varepsilon)^{-1}(0)$ and $(\widehat{f})^{-1}(0) = \mathrm{SOL}(f, K)$, we have $x(\varepsilon) \in \mathrm{SOL}(f, K) + \zeta B$. Hence for all such ε we have

$$dist\,(x(\varepsilon), \mathrm{SOL}(f, K)) \leq \zeta.$$

This implies (b).
(c) $(i) \Longrightarrow (ii)$: We suppose (i) holds. Then by definition, $\mathrm{SOL}(f, K)$ is nonempty. To see (ii),

that is, to see the boundedness of $\mathrm{SOL}(f,K)$, we follow an argument given in the proof of Theorem 4.4, [7]. Assume that $\mathrm{SOL}(f,K)$ is unbounded. We produce small perturbations $f(x)+\delta\eta(x)$ for which the corresponding BVI has no solution. Let $\{x^k\}$ be an unbounded sequence of solutions in $\mathrm{SOL}(f,K)$. Without loss of generality, we may take a nonempty index set $I\cup J$ and a vector $\bar{x}$ such that $x_i^k \to \infty\ (i\in I)$, $x_i^k \to -\infty\ (i \in J)$, and $x_i^k \to \bar{x}_i\ (i \notin I\cup J)$. Let η be a function defined by

$$\eta_i(x) := \begin{cases} -e^{-x_i} & \text{if } i\in I \\ e^{x_i} & \text{if } i \in J \\ (x_i-\bar{x}_i) & \text{if } i\notin I\cup J \end{cases}$$

Since $f(x)+\delta\eta(x)$ is a **P**-function (for $\delta>0$), by (i), $\mathrm{SOL}(f+\delta\eta,K)$ is singleton for all small $\delta>0$. For any such δ, by Corollary 3 applied to the function $x\mapsto x-\Pi_K[x-f(x)-\delta\eta(x)]$, for all q sufficiently close to zero, the sets

$$\{x:\ x-\Pi_K[x-f(x)-\delta\eta(x)]=q\}$$

are uniformly bounded. We show that this is false by showing

$$x^k-\Pi_K[x^k-f(x^k)-\delta\eta(x^k)]\to 0 \tag{15}$$

as $k\to 0$. Now (15) follows easily from $x^k-\Pi_K[x^k-f(x^k)]=0$, $\eta(x^k)\to 0$ and the inequality

$$||\{x^k-\Pi_K[x^k-f(x^k)-\delta\eta(x^k)]\}-\{x^k-\Pi_K[x^k-f(x^k)]\}||\leq \delta||\eta(x^k)||$$

Thus we reach a contradiction. Hence $(i)\implies(ii)$.
$(ii)\implies(iii)$, $(ii)\implies(i)$: Assume that $\mathrm{SOL}(f,K)$ is nonempty and bounded and let $\varepsilon>0$. Then by Remark 2 in Section 2 of [12], $\deg(\widehat{f},\Omega,0)=\pm 1$ for $\Omega:=\mathrm{SOL}(f,K)+\varepsilon\mathcal{B}$. It follows from the nearness property of degree (Thm. 2.1.2, [15]) that for any continuous function g with

$$\sup_{\Omega}||g(x)-f(x)||<\ \widehat{\delta}:=\mathrm{dist}\,(0,\widehat{f}(\partial\Omega))$$

$\deg(\widehat{g},\Omega,0)=\pm1$ where $\widehat{g}(x)=x-\Pi_K(x-g(x))$. Hence $\widehat{g}$ will have a zero in Ω. By taking $\delta^*=\widehat{\delta}(1+\sup_\Omega||x||)^{-1}$, we verify the linear stability of $\mathrm{BVI}(f,K)$.
For a given continuous h and $\varepsilon>0$, we take a $\overline{\delta}>0$ such that $\overline{\delta}(\sup_\Omega||h(x)||)<\widehat{\delta}$ and verify the directional stability of $\mathrm{BVI}(f,K)$.

Now suppose that each interval K_i is bounded below. Suppose we have (iii) and that the solution set is unbounded. We proceed as in the proof of $(i)\implies(ii)$. Since in this setting, the index set J is empty, we see that resulting function $f(x)+\delta\eta(x)$ satisfies the linear stability condition for small δ. As before we get a contradiction when $\mathrm{BVI}(f+\delta\eta,K)$ has a solution for small δ. This completes the proof. $\square$

By specializing K and μ, one could get various special cases of the above theorem. In particular, by taking $K = R^n_+$, and μ as the point mass at the origin, we get the following generalization of Facchinei and Kanzow results (mentioned in the Introduction) for continuous $\mathbf{P}_0$-functions in the NCP setting.

Corollary 6 *Consider* $\mathrm{NCP}(f)$ *where* f *is a continuous* $\mathbf{P}_0$*-function. Let* $f_\varepsilon(x) = f(x) + \varepsilon x$*. Then the following hold.*

(a) For each $\varepsilon > 0$, $\mathrm{NCP}(f_\varepsilon)$ *has a unique solution* $x(\varepsilon)$ *and moreover, the mapping* $\varepsilon \mapsto x(\varepsilon)$ *is continuous on* $(0, \infty)$.

(b) If $\mathrm{SOL}(f)$ *is nonempty and bounded, then dist* $(x(\varepsilon), \mathrm{SOL}(f)) \to 0$ *as* $\varepsilon \to 0$.

(c) $\mathrm{NCP}(f)$ *is linearly stable if and only if* $\mathrm{SOL}(f)$ *is nonempty and bounded.*

4 Appendix

Lemma 1 *Consider four real numbers* $\alpha,\ \beta,\ \gamma,\ \ \delta$ *with* $\alpha > \beta$ *and* $\gamma > \delta$*. Let* L *be a closed interval in* R*. Then*

$$\alpha - \Pi_L(\alpha - \gamma) > \beta - \Pi_L(\beta - \delta).$$

Note: With a limiting argument, one can see that $\alpha > \beta, \gamma \geq \delta \Longrightarrow \alpha - \Pi_L(\alpha - \gamma) \geq \beta - \Pi_L(\beta - \delta)$.

Proof. Suppose if possible, $\alpha - \Pi_L(\alpha - \gamma) \leq \beta - \Pi_L(\beta - \delta)$. We consider all possible cases and show that in each case, the inequality fails. The possible values of $\alpha - \Pi_L(\alpha - \gamma)$ are

(1) $\alpha - l \;\; if \;\; \alpha - \gamma \leq l$ where $l = \inf L > -\infty$

(2) $\gamma \;\; if \;\; l \leq \alpha - \gamma \leq u$ where $l = \inf L > -\infty$ and $u = \sup L < \infty$

(3) $\alpha - u \;\; if \;\; u \leq \alpha - \gamma$ where $u = \sup L < \infty$

and the possible values of $\beta - \Pi_L(\beta - \delta)$ are

(a) $\beta - l \;\; if \;\; \beta - \delta \leq l$ where $l = \inf K > -\infty$

(b) $\delta \;\; if \;\; l \leq \beta - \delta \leq u$ where $l = \inf L > -\infty$ and $u = \sup L < \infty$

(c) $\beta - u \;\; if \;\; u \leq \;\; \beta - \delta$ where $u = \sup L < \infty$.

We look at the following cases under the assumption that $\alpha - \Pi_L(\alpha - \gamma) \leq \beta - \Pi_L(\beta - \delta)$.

(i) (1) and (a) hold: This is not possible since $\alpha > \beta$.

(ii) (2) and (b) hold: This is not possible since $\gamma > \delta$.

(iii) (3) and (c) holds: This is not possible since $\alpha > \beta$.

(iv) (1) and (b) hold: Then $\alpha - l \leq \delta$. Since $\alpha - l > \beta - l$, this implies $\beta - l < \delta$ which contradicts (b).

(v) (1) and (c) hold: Then $\alpha - u \leq \alpha - l \leq \beta - u$ contradicts $\alpha > \beta$.

(vi) (2) and (a) hold: Then $\gamma \leq \beta - l \leq \delta$ which contradicts $\gamma > \delta$.

(vii) (2) and (c) hold: Then $\gamma \leq \beta - u < \alpha - u$ contradicts (2).

(viii) (3) and (a) hold: Then $\gamma \leq \alpha - u \leq \beta - l \leq \delta$ contradicts $\gamma > \delta$.

(ix) (3) and (b) hold: Then $\alpha - u \leq \delta < \gamma$ contradicts (3).

This completes the proof of the Lemma. □

Concluding remarks In this paper, based on a result of Banach and Mazur, and on degree theory, we have generalized some results of Facchinei and Kanzow. These generalizations deal with integral regularizations of the fixed point map corresponding to a box variational inequality problem. The ideas of the paper can be used in other contexts as well. For example, one could study regularizations based on the normal map and on smoothing. Such a study will be carried out in a separate paper.

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