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On Bit-Error Probability of a Concatenated Coding Scheme

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***Abstract*— This paper presents a method for evaluating the bit-error probability of a concatenated coding system for BPSK transmission over the AWGN channel. In the concatenated system, a linear binary block code is used as the inner code and is decoded with the soft-decision maximum likelihood decoding, and a maximum distance separable code (or its interleaved code) is used as the outer code and is decoded with a bounded distance decoding. The method is illustrated through a specific example in which the inner code is a binary $(64, 40, 8)$ Reed–Muller subcode and the outer code is the NASA standard $(255, 223, 33)$ Reed–Solomon code over GF(2^8) interleaved to a depth of 5. This specific concatenated system is being considered for NASA's high-speed satellite communications. The bit-error performance is evaluated by a combination of simulation and analysis. The split weight enumerators for the maximum distance separable codes are derived and used for the analysis.**

I. Introduction

Concatenated coding [1] is a technique of combining relatively simple codes to form a powerful coding system for achieving high performance (or very low error probability) and large coding gain with reduced decoding complexity. Fig. 1 depicts a single-level concatenated coding system in which an outer code and an inner code are combined in tandem (or cascade). In practical applications, the inner code is usually a relatively short binary block code or a binary convolutional code of relatively short constraint length (or small memory size), and the outer code is usually a Reed–Solomon code with symbols from a Galois field $\mathrm{GF}(2^m)$. Encoding is accomplished in two steps, first the outer code encoding and then the inner code encoding. Decoding is carried out in two stages, the inner code decoding and the outer code decoding. This two-stage decoding simplifies the decoding complexity. The inner code decoding can be either soft-decision decoding or hard-decision decoding. Outer code decoding is usually carried out in hard-decision to reduce decoding complexity. If the inner and outer codes are properly chosen, the combination of soft-decision inner code decoding and hard-decision outer code decoding achieves high performance and large coding gain with only moderate decoding complexity.

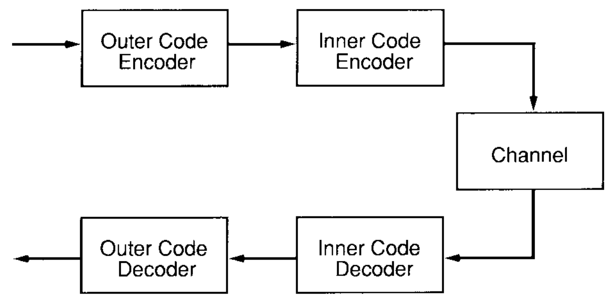

Fig. 1. Single-level concatenated coding system.

In this paper, we investigate the bit-error performance of a class of single-level concatenated coding systems with BPSK transmission over the AWGN channel. In each system, the inner code is a binary linear block code, and the outer code is a maximum distance separable (MDS) code (or its interleaved code). The inner code is decoded with the soft-decision maximum likelihood decoding (MLD), and the outer code is decoded with hard-decision bounded distance decoding. Recent study shows that linear block codes do have a trellis structure, and they can be decoded with the Viterbi algorithm [2]–[3]. Some well-known linear block codes, such as Reed–Muller codes, have very simple trellis diagrams which are quite suitable for high-speed decoding [4]–[8].

Block error performance of a single-level concatenated coding system with soft-decision inner code decoding was analyzed [9], in which the bit error probability was roughly approximated. In many practical applications, the bit-error probability is a better measure of the system performance than the block-error probability. Consequently, a more precise evaluation of the bit-error probability of a coding system is needed. In this paper, we present a method for analyzing and evaluating the bit-error probability in the information part of a single-level concatenated coding system with soft-decision MLD for the inner code as described above. The analysis is carried out based on the split-weight spectrum of the outer code, a maximum distance separable code. The method is illustrated by a specific single-level concatenated coding system.

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The paper is organized as follows. The second section presents a specific single-level concatenated coding system with which we illustrate our method for analyzing the bit-error probability. This system is being considered by NASA for high-speed satellite communications. The inner code of this system is a subcode of a Reed–Muller (RM) code, and the outer code is a Reed–Solomon (RS) code. The inner code is to be decoded with the (soft-decision) Viterbi algorithm. Section III presents the trellis structure and complexity of the inner code. The analysis of the bit-error performance of the system is given in Section IV. The split-weight distribution of the outer code is used in evaluating the bit-error probability. An improvement of the proposed concatenated coding system is presented in Section V. The split-weight enumerators for the maximum distance separable codes, which include RS codes as a subclass, are derived in the Appendix.

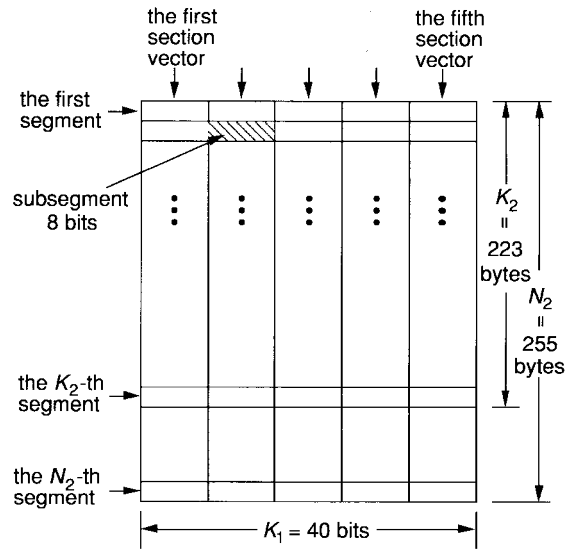

Fig. 2. A decoding array.

II. A Single-Level Concatenated Coding System

As we pointed out earlier, the purpose of this paper is to present a method to evaluate the bit-error performance of a class of single-level concatenated coding systems. To present the method, we use a specific single-level concatenated coding system as a working example.

Let (N, K, D) denote a linear block code of length N, dimension K, and minimum Hamming distance D. Let $\mathrm{RM}_{6,3}$ denote the third-order RM code of length 2^6. This RM code is a $(64, 42, 8)$ code. In the proposed concatenated system, the inner code, denoted C_1, is an $(N_1, K_1, D_1) = (64, 40, 8)$ subcode of the RM code, $\mathrm{RM}_{6,3}$. For convenience, we call it an RM subcode. This RM subcode has a relatively simple trellis structure, and hence can be decoded with the Viterbi algorithm to reduce decoding complexity. The outer code of the proposed concatenated coding system, denoted C_2, is the NASA standard $(N_2, K_2, D_2) = (255, 223, 33)$ RS code over $\mathrm{GF}(2^8)$. This RS outer code is interleaved with a depth (or degree) of $m = 5$. Each code symbol of this outer code is represented by a binary 8-tuple, called a byte, based on a certain basis of $\mathrm{GF}(2^8)$. Using this representation, a codeword in the RS outer code consists of 255 8-bit bytes, or 255×8=2040 bits.

The encoding of the proposed concatenated coding scheme is accomplished in two steps, the outer code encoding and the inner code encoding. First, a message of K_2 bytes (or K_2×8 bits) is encoded into a codeword of N_2 bytes in the outer code C_2. This codeword is then stored in a buffer as a column N_2 bytes long. After five outer codewords have been formed, the buffer stores a 255×5 array over $\mathrm{GF}(2^8)$. Each row consists of 5 bytes (40 bits). At the second stage of encoding, each row is encoded into a codeword of 64 bits (or 8 bytes) in the inner code which is mapped into a sequence of 64 BPSK signals and transmitted.

The decoding also consists of two stages. Every received sequence of 64 signals is decoded into an inner code codeword. The inner code is decoded with the soft-decision MLD using the Viterbi algorithm. After each inner code decoding, the decoded information bits (5 bytes) are stored in a receiver buffer as a row of an array as shown in Fig. 2. This row is called a segment, which consists of 5 bytes; each byte is called a subsegment, which represents a symbol in $\mathrm{GF}(2^8)$. Each column is called a section, which consists of 255 bytes [or symbols in $\mathrm{GF}(2^8)$]. At the second stage of decoding, each column of the array is decoded based on the $(255, 223, 33)$ RS outer code. The RS outer code is capable of correcting $t_2 = 16$ symbol errors. If the syndrome of a column corresponds to an error pattern of t_2 or fewer symbol errors, error correction is performed, and the decoded information symbols are then delivered to the user. If more than t_2 symbol errors are detected, the outer code decoder stops the decoding of the column, and outputs the symbols in the information part of the column to the user.

In the following, we analyze the bit-error performance of the above concatenated coding system.

III. Trellis Structure and Complexity of the Inner Code

The inner code C_1 of the proposed concatenated coding system is a specific subcode of the RM code, $\mathrm{RM}_{6,3}$. In terms of Boolean polynomials [11], the basis of this RM subcode consists of vectors corresponding to monomials (single-term Boolean polynomials) of degree 3 or less except $x_4x_5x_6$ and $x_3x_5x_6$. C_1 has a relatively simple trellis structure. For $L = 4$ and 8, the measures of structural complexities of the L-section minimal trellis diagrams are given in Table I. The structural complexity of a trellis diagram is measured in terms of state complexity, branch complexity, state connectivity, and parallel structure [5]–[8]. A trellis diagram T is said to be reversible if the graph T_R, obtained from T by reversing the direction of each branch and its label and exchanging the initial state and the final state, is identical to T. For a reversible trellis, the left half and the right half of the trellis are structurally identical (i.e., they are mirror image of each other). This mirror symmetry allows bi-directional decoding of the code. The

TABLE I
MEASURES OF STRUCTURAL COMPLEXITY OF PARALLEL COMPONENTS OF THE L-SECTION MINIMAL TRELLIS DIAGRAM WITH $L = 4$ AND 8 FOR THE $(64, 40, 8)$ SUBCODE OBTAINED FROM $\mathrm{RM}_{6,3}$

L	4		8			
i	1	2	1	2	3	4
$K_{(i-1)\Lambda}$	0	8	0	6	8	11
$K_{i\Lambda}$	8	8	6	8	11	8
$K_{(i-1)\Lambda,i\Lambda}$	5		1			
$Q_{(i-1)\Lambda,i\Lambda}$	0	6	0	3	3	6

1) Λ denotes N_1/L.
2) The number of states at the end of the ith section (or just after the Λith bit) is $2^{K_{\Lambda i}}$.
3) For each state s at the Λith bit, there are $2^{Q_{\Lambda(i-1),\Lambda i}}$ states at the $\Lambda(i-1)$th bit from which there are branches to s, and the number of parallel branches is $2^{K_{\Lambda(i-1),\Lambda i}}$.

TABLE II
THE COMPLEXITY MEASURES OF SOFT-DECISIONMAXIMUM LIKELIHOOD DECODING FOR THE (64, 40) CODE OBTAINED FROM $\mathrm{RM}_{6,3}$ WHEN AN L-SECTION MINIMAL TRELLIS DIAGRAM IS USED

L	4	8	16	32	64
AD_{BM}	15360	7168	768	128	0
CP_{PB}	23552	512	0	0	0
AD	33024	69696	87312	92068	138746
CP	32511	64767	73599	56191	56191

minimal L-section trellis diagrams for the inner code C_1 are reversible. Therefore, in Table I, we only list the complexity measures for the left half of the L-section trellis diagrams with $L = 4$ and 8.

C_1 has relatively simple trellis structure, since $\mathrm{RM}_{6,3}$ does. The four-section minimal trellis diagram for $\mathrm{RM}_{6,3}$ consists of 16 parallel and structurally identical (except branch labels) 64-state subtrellis diagrams without cross connections between them. The four-section minimal trellis diagram for C_1 consists of four of the subtrellis diagrams. Hence, the number of states (or branches) in a minimal four-section trellis diagram for the inner code C_1 is one-fourth of that for its supercode, $\mathrm{RM}_{6,3}$. This parallel structure allows us to devise four identical 64-state Viterbi decoders to process the decoding in parallel. This not only simplifies the decoding complexity, but also speeds up the decoding process.

Consider the complexity of Viterbi decoding for the $(64, 40, 8)$ inner code C_1 based on an L-section minimal trellis diagram. The complexity is measured by: (M1) the total number $\mathrm{AD}_{\mathrm{BM}}$ of additions for the branch metric computations, (M2) the total number $\mathrm{CP}_{\mathrm{PB}}$ of comparisons to find the largest branch metric among each set of parallel branches, (M3) the total number AD of additions of the largest branch metric among those parallel branches and the survivor's metric of the state from which the branches diverge, and (M4) the total number CP of comparisons to find a survivor at each state [8].

In the ith section of an L-section trellis, let 2^{b_i} denote the number of distinct branches. If the branch metric computations in the ith section are done in the most parallel manner, a total of at most $2^{b_i}(\Lambda - 1)$, where $\Lambda \triangleq N_1/L$, additions are required. There are slower methods of computing the same set of branch metrics that result in smaller number of additions. To find the largest branch metric among each distinct set of 2^{p_i} parallel branches, $(2^{p_i} - 1)$ comparisons are required. The values of $\mathrm{AD}_{\mathrm{BM}}$ and $\mathrm{CP}_{\mathrm{PB}}$ are evaluated for the $(64, 40)$ code with $L \geq 8$ in this manner. For $L = 4$, each set of parallel branches is a coset of the first-order RM code of length 16. By using this structure, we can reduce the number of additions and comparisons without slowing down the decoding. The two-section minimal trellis diagram of the coset has eight states just after the eighth bit position, and for each of the states, there are two parallel branches from the initial states and two parallel branches to the final state. When the two-section trellis is used to find the the largest branch metric for each set of 2^5 parallel branches, the value of $\mathrm{AD}_{\mathrm{BM}}$ is given by

$$\mathrm{AD}_{\mathrm{BM}} = 7168 + 4 \cdot 2^8 \cdot 8 = 15\,360.$$

The number of additions to compute the branch metrices for all the two-section trellises is equal to the value of $\mathrm{AD}_{\mathrm{BM}}$ with the $L = 8$, and is 7168 (refer to Table II). Eight additions are required for each two-section trellis. Also, $\mathrm{CP}_{\mathrm{PB}}$ is given by

$$\mathrm{CP}_{\mathrm{PB}} = 4 \cdot 2^8 \cdot 23 = 23\,552.$$

Hence, $\mathrm{AD}_{\mathrm{BM}}$ and $\mathrm{CP}_{\mathrm{PB}}$ are reduced to 1/32 and 23/31, respectively. Based on this improved method, the values of $\mathrm{AD}_{\mathrm{BM}}$ and $\mathrm{CP}_{\mathrm{PB}}$ for the (64, 40, 8) inner code with $L = 4$ are evaluated and given in Table II. The number of states at the end of the ith section (or just after the $i\Lambda$th bit in the L-section trellis diagram, is denoted by $2^{K_{i\Lambda}}$. For each state s at the $i\Lambda$th bit, those states at the $(i-1)\Lambda$th bit from which there are branches to s are denoted by $2^{Q_{(i-1)\Lambda,i\Lambda}}$ (refer to Table I). Then

$$\mathrm{AD} = \sum_{i=2}^{L} 2^{Q_{(i-1)\Lambda,i\Lambda} + K_{i\Lambda}} \tag{1}$$

$$\mathrm{CP} = \sum_{i=2}^{L} (2^{Q_{(i-1)\Lambda,i\Lambda}} - 1) 2^{K_{i\Lambda}}. \tag{2}$$

Note that in this definition of AD, the summation is taken from $i = 2$ to L, since the metric of the initial state is 0 and therefore the additions in the first section are not necessary.

The values of $\mathrm{AD}_{\mathrm{BM}}$, $\mathrm{CP}_{\mathrm{PB}}$, AD, and CP for the 2^j-section minimal trellis diagrams with $2 \leq j \leq 6$ are listed in Table II.

IV. ERROR PERFORMANCE ANALYSIS

A. Error Performance Analysis of the Inner Code

We analyze the error performance of the example coding scheme described in Section II, assuming that all codewords

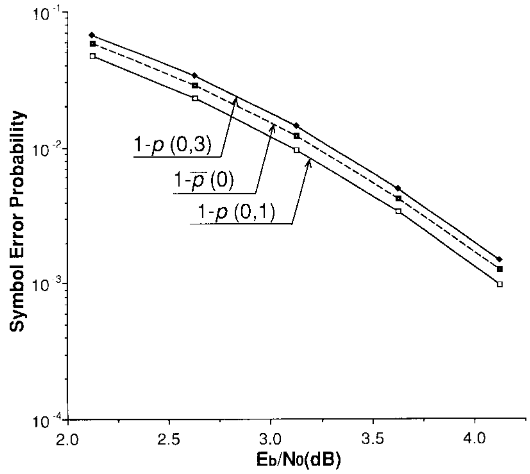

Fig. 3. Simulation results on the symbol error probability of the most erroneous subsegment and that of the least one, and the average of the subsegment error probabilities over the five subsegments.

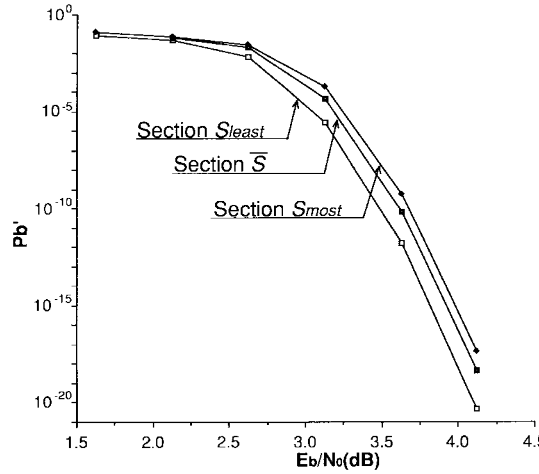

Fig. 4. Calculation results on the approximations of the bit-error probabilities, P_b' for sections, $S_{\rm most}$, (i.e, $j=3$), $S_{\rm least}$ (i.e. $j=1$)and $\overline{S}$.

are generated equally likely. Hereafter, we assume that the all-zero word is transmitted for simplicity.

The symbol error probability for the decoded segment is evaluated by simulation. For $1 \le j \le m$(the interleaving depth) and $a \in {\rm GF}(q)$, let $p(a,j)$ denote the probability that the jth subsegment of the decoded segment is decoded into a by the inner decoder. Let $\bar{p}(a)$ denote the average of $p(a,j)$ over the five subsegments.

Fig. 3 shows simulation results on the symbol error probability of the most erroneous subsegment, $1-p(0,3)$, that of the least one, $1-p(0,1)$, and the average of the symbol error probabilities over the five subsegments, $1-\bar{p}(0)$. An information bit assignment of the inner (64, 40) code for the interleaved outer code symbol (i.e., how to divide 40 information bits of the inner code into five 8-bit symbols of outer codewords) affects the symbol error probability of each symbol. As we can see in the figure, the differences are among the symbol error probabilities are not small. However, because the bit errors of each information bit on the inner decoding depend on each other, it is hard to find a good information bit assignment. A solution is to take the $(1+(i+j) \bmod 5)$th subsegment of the ith segment as the ith symbol of the jth section vector.

B. Error Performance Analysis of the Outer Code

It is difficult to analyze the error performance of a concatenated code with relatively large parameters by using the conventional simulation algorithm because the total probability of an incorrect decoding and a decoding failure of the outer code drops drastically as the signal-to-noise ratio (SNR) increases. Hence we need a more efficient algorithm for evaluating the bit-error probability on the information part of the outer code.

In our analysis, the probability of an incorrect decoding for an outer decoding is much less than that of a decoding failure for all the ranges of SNR in which we are interested.

Accordingly, we first consider an approximation of the bit-error probability of the coding scheme in the jth section, denoted $P_b'(j)$. For $s \in {\rm GF}(q)$, let $wt(s)$ denote the weight of the binary representation of s. Let R_2 denote $N_2 - K_2$ and l denote the number of bits for a symbol.

For $1 \le j \le m$, define $P_b'(j)$ as follows:

$$P_b'(j) \triangleq \frac{\tilde{w}_E(j)}{\ell K_2} \sum_{\substack{0 \le i_1 \le K_2 \\ 0 \le i_2 \le R_2 \\ t_2 < i_1 + i_2 \le N_2}} \left\{ i_1 \binom{K_2}{i_1} \binom{R_2}{i_2} \times (1-p(0,j))^{i_1+i_2} p(0,j)^{N_2-i_1-i_2} \right\} \quad (3)$$

where

$$\tilde{w}_E(j) = \sum_{s \in {\rm GF}(q)} wt(s)p(s,j)/(1-p(0,j)). \quad (4)$$

$P_b'(j)$ is actually the bit-error probability in the jth section in the virtual case that any section vector within Hamming distance t_2 or less from the transmitted codeword is decoded correctly, but other section vectors, which are at Hamming distance t_2+1 or greater from the transmitted codeword, are unsuccessfully decoded and only pass through the outer decoder without any correction.

Let $S_{\rm most}$ denote the section for which the symbol error probability is the largest among all the sections and $S_{\rm least}$ denote the section for which the symbol error probability is the least among all the sections. For the example scheme, $S_{\rm most} = 3$ and $S_{\rm least} = 1$. Let $\bar{S}$ denote a conceptual section in which the probability that the zero symbol is decoded into $a \in {\rm GF}(q)$ by the inner decoder is $\bar{p}(a)$. Fig. 4 shows the computation results of the $P_b'(j)$ for the sections $S_{\rm most}$, $S_{\rm least}$ and $\bar{S}$ of the example scheme.

Let $P_b(j)$ be defined as the bit-error probability of the outer decoder for the jth section, and let the effect of incorrect decoding on the bit-error probability be defined as $P_b(j) - P_b'(j)$. In the following, we show how to evaluate the difference $P_b(j) - P_b'(j)$.

Suppose that a section vector $\bar{\boldsymbol{\epsilon}} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{N_2})$ is decoded into a codeword $\boldsymbol{v} = (v_1, v_2, \ldots, v_{N_2}) \ne \mathbf{0}$ by the outer decoder. Now, let $I(\boldsymbol{v}, \bar{\boldsymbol{\epsilon}})$ denote the set of indexes i

for which $v_i \neq \epsilon_i$, where $|I(\boldsymbol{v},\bar{\boldsymbol{\epsilon}})| \leq t_2$. Define $I_K(\boldsymbol{v},\boldsymbol{\epsilon})$ as the intersection $I(\boldsymbol{v},\bar{\epsilon}) \cap \{1,2,\ldots,K_2\}$. Then the effect of decoding $\bar{\boldsymbol{\epsilon}}$ into $\boldsymbol{v}$ on the bit-error probability for the jth section can be computed as follows:

$$\frac{P(\bar{\boldsymbol{\epsilon}},j)}{\ell K_2}\sum_{i\in I_K(\boldsymbol{v},\bar{\boldsymbol{\epsilon}})}(wt(v_i)-wt(\epsilon_i)) \tag{5}$$

where $P(\bar{\boldsymbol{\epsilon}},j)$ denotes the occurrence probability of $\bar{\boldsymbol{\epsilon}}$ in the jth section.

Let IS be defined as $\{\bar{I}:\bar{I}\subset\{1,2,\ldots,N_2\}$ and $|\bar{I}|\leq t_2\}$. For $\boldsymbol{v}$ and $\bar{I}\in IS$, let $ES(\boldsymbol{v},\bar{I})$ be defined as

$$ES(\boldsymbol{v},\bar{I})=\{\bar{\boldsymbol{\epsilon}}\in \mathrm{GF}(q)^{N_2}:\bar{I}=I(\boldsymbol{v},\bar{\boldsymbol{\epsilon}})\}. \tag{6}$$

Let $P_{bd}(\boldsymbol{v},j)$ denote the effect of all section vectors that are decoded into $\boldsymbol{v}$ incorrectly on the bit-error probability in the jth section. Then $P_{bd}(\boldsymbol{v},j)$ is given as follows

$$P_{bd}(\boldsymbol{v},j)=\frac{1}{\ell K_2}\sum_{\bar{I}\in IS}\sum_{\bar{\boldsymbol{\epsilon}}\in ES(\boldsymbol{v},\bar{I})}\times\left\{P(\bar{\epsilon},j)\sum_{i\in I_K(\boldsymbol{v},\bar{\epsilon})}(wt(v_i)-wt(\epsilon_i))\right\} \tag{7}$$

where $P(\bar{\boldsymbol{\epsilon}},j)=\prod_{h=1}^{N_2}p(\epsilon_h,j)$. For $1\leq i\leq K_2$ and $s\in\mathrm{GF}(q)-\{v_i\}$, let $IS_i \triangleq \{\bar{I}':\bar{I}'\subset\{1,\ldots,N_2\}-\{i\}$ and $|\bar{I}'|<t_2\}$, and for $\bar{I}'\in IS_i$, let $ES_i(\boldsymbol{v},\bar{I}',s)\triangleq\{\bar{\boldsymbol{\epsilon}}\in ES(\boldsymbol{v},\bar{I}'\subset\{i\}):\epsilon_i=s\}$. Then, the right-hand side of (7) can be rearranged as follows

$$\begin{aligned}P_{bd}(\boldsymbol{v},j)&=\frac{1}{\ell K_2}\sum_{i=1}^{K_2}\sum_{s\in\mathrm{GF}(q)-\{v_i\}}\sum_{\bar{I}'\in IS_i}\sum_{\bar{\epsilon}\in ES_i(\boldsymbol{v},\bar{I}',s)}\\&\quad\times(wt(v_i)-wt(s))p(s,j)\prod_{\substack{h=1\\h\neq i}}^{N_2}p(\epsilon_h,j)\\&=\frac{1}{\ell K_2}\sum_{i=1}^{K_2}\left\{wt(v_i)(1-p(v_i,j))\right.\\&\quad\left.-\sum_{\substack{s\in\mathrm{GF}(q)\\s\neq v_i}}wt(s)p(s,j)\right\}Q_i(\boldsymbol{v},j)\end{aligned} \tag{8}$$

where

$$Q_i(\boldsymbol{v},j)=\sum_{\bar{I}'\in IS_i}\left\{\prod_{k\in\bar{I}'}(1-p(v_k,j))\prod_{\substack{k\in\{1,\ldots,N_2\}\\-\{i\}-\bar{I}'}}p(v_k,j)\right\}. \tag{9}$$

We now consider how to compute $Q_i(\boldsymbol{v},j)$ efficiently. For $0<i_1\leq i_2\leq N_2$ and $0\leq k<\min\{t_2,i_2-i_1+2\}$, let $P_{i_1,i_2}(\boldsymbol{v},k,j)$ denote the probability that the Hamming distance between a section vector and $\boldsymbol{v}$ from the i_1th component to the i_2th component is k. Then, $P_{i_1,i_2}(\boldsymbol{v},k,j)$ is given as follows:

$$P_{i,i}(\boldsymbol{v},0,j)\triangleq p(v_i,j)\qquad \text{for}\ \ 1\leq i\leq N_2 \tag{10}$$

$$P_{i,i}(\boldsymbol{v},1,j)\triangleq 1-p(v_i,j)\qquad \text{for}\ \ 1\leq i\leq N_2 \tag{11}$$

$$\begin{aligned}P_{i_1,i_2}(\boldsymbol{v},k,j)&\triangleq P_{i_1,i_2-1}(\boldsymbol{v},k,j)P_{i_2,i_2}(\boldsymbol{v},0,j)\\&\quad+P_{i_1,i_2-1}(\boldsymbol{v},k-1,j)P_{i_2,i_2}(\boldsymbol{v},1,j)\\&\qquad\text{for}\ \ 0<i_1<i_2\leq N_2.\end{aligned} \tag{12}$$

For $1\leq i\leq K_2, Q_i(\boldsymbol{v},j)$ is expressed as

$$\begin{aligned}Q_i(\boldsymbol{v},j)&=\sum_{\substack{0\leq k_1+k_2<t_2\\0\leq k_1<i\\0\leq k_2}}P_{1,i-1}(\boldsymbol{v},k_1,j)P_{i+1,N_2}(\boldsymbol{v},k_2,j)\\&\qquad\qquad\text{for } 1<i\leq K_2\\&=\sum_{k=0}^{t_2-1}P_{2,N}(\boldsymbol{v},k,j)\qquad\text{for } i=1.\end{aligned} \tag{13}$$

Consequently, (8) can be computed efficiently as follows

$$\begin{aligned}P_{bd}(\boldsymbol{v},j)&=\frac{1}{\ell K_2}\sum_{i=1}^{K_2}\left\{wt(v_i)(1-p(v_i,j))\right.\\&\quad\left.-\sum_{\substack{s\in\mathrm{GF}(q)\\s\neq v_i}}wt(s)p(s,j))\right\}Q_i(\boldsymbol{v},j)\\&=\frac{1}{\ell K_2}\sum_{i=0}^{K_2}\{wt(v_i)-\tilde{w}_E(j)\,(1-p(0,j))\}Q_i(\boldsymbol{v},j).\end{aligned} \tag{14}$$

This probability will be used in computing $P_b(j)-P_b'(j)$.

To compute $P_b(j)-P_b'(j)$, we need to generate the codewords of the outer MDS code and compute their split weight spectrum. To generate the codewords of the outer code, we use the following known fact of MDS codes [13]. Let the support of a codeword $\mathbf{v}$ be defined as the set of indexes of nonzero components of $\mathbf{v}$.

Fact–MDS: For a given set $I^{(2)}$ of R_2 indexes in $\{1,2,\ldots,N_2\}$ and a given index $i\in\{1,2,\ldots,R_2\}\backslash I^{(2)}$, it is easy to compute $(\beta_1,\beta_2,\ldots,\beta_{R_2})\in GF(q)^{R_2}$ such that the codeword, whose support is $\{i\}\cup I^{(2)}$ and whose ith component is nonzero $\alpha\in GF(q)$, has $\alpha\beta_{i'}$ as the i'th component for $i'\in I^{(2)}$. Let this codeword be denoted by $\alpha g_{I^{(2)}}(i)$. For a set $I^{(1)}$ of indexes in $\{1,2,\ldots,N_2\}\backslash I^{(2)}$ and nonzero $\alpha_i\in GF(q)$ with $i\in I^{(1)}$, the codeword, whose support is a union of I^1 and a subset of $I^{(2)}$ and whose ith component is α, for $i\in I^{(1)}$, is $\Sigma_{i\in I^{(1)}}\alpha_i g_{I^{(2)}}(i)$. △△

For $1\leq w_1\leq K_2$ and $1\leq w_2\leq R_2$ such that $w_1+w_2>R_2$, let C_{2,w_1,w_2} denote the set of outer codewords with weight w_1 in the information part and weight w_2 in the redundancy part. Based on Fact-MDS, codewords in C_{2,w_1,w_2} can be generated efficiently for a low weight w_1+w_2 as follows:

(**G1**) Choose a set I_I of w_1 indexes in $\{1,2,\ldots,K_2\}$ randomly, and choose a set I_R of w_2 indexes in $\{K_2+1,K_2+2,\ldots,N_2\}$.

(**G2**) Let $I^{(1)}$ be the set of $w_1+w_2-R_2$ smallest indexes of $I_I \cup I_R$, and let $I^{(2)}$ denote $(I_I \cup I_R) - I^{(1)}$. For every nonzero $\alpha_i \in GF(q)$ with $i \in I^{(1)}$, compute $\mathbf{v} \triangleq \Sigma_{i \in I^{(1)}} \alpha_i g_{I^{(2)}}(i)$. If the weight of $\mathbf{v}$ is less than $w_1 + w_2$ (the possibility is small), then discard $\mathbf{v}$.

The support of codewords with a given low weight generated by this procedure are chosen randomly.

For $1 \leq w_1 \leq K_2, 1 \leq w_2 \leq R_2$ and $w_1 + w_2 > R_2$, we compute $P_{bd}(\mathbf{v}, j)$ for each randomly generated outer codeword $\mathbf{v}$ in C_{2,w_1,w_2} and fake their average, denoted $P_{bd_{w_1,w_2}}(j)$.

Let A_{w_1,w_2} denote the number of outer codewords that have w_1 nonzero symbols in the information part and w_2 nonzero symbols in the redundancy part. Then the effect of incorrect decoding on the bit-error probability, $P_b(j) - P'_b(j)$, is given by the following formula:

$$\sum_{i_1=1}^{K_2} \sum_{i_2=N_2-K_2+1-i_1}^{N_2-K_2} A_{i_1,i_2} P_{bd_{i_1,i_2}}(j) \tag{15}$$

Simulation results on $P_b(j) - P'_b(j)$ for sections S_{most} and S_{least} of the example scheme are shown in Fig. 5.

The split weight enumerators for MDS codes are derived in the Appendix.

For a positive integer n, let $\tilde{P}^{(n)}_{bd_{w_1,w_2}}(j)$ denote the average of $P_{bd}(\mathbf{v}, j)$ over the codewords of C_{2,w_1,w_2} generated by (G1) and (G2), whose supports are generated by one of the first n trials in (G1). The relative deviation of $\tilde{P}^{(n)}_{bd_{w_1,w_2}}(j)$ became very small after 500 to 2000 trials for the range of SNR shown in Fig. 5. We actually made 5000 to 20 000 trials. Let $\tilde{n}_{w_1,w_2}$ denote the number of random trials for weights w_1 and w_2. Then the number of generated codewords in C_{2,w_1,w_2} is about $q^{w_1+w_2-R_2}\tilde{n}_{w_1,w_2}$. The larger $w_1+w_2-R_2$ and the higher SNR are, the faster the above convergence is (that is, the smaller $\tilde{n}_{w_1,w_2}$ is sufficient). As $w_1 + w_2 - R_2$ grows, $A_{w_1,w_2}\tilde{P}^{(\tilde{n}_{w_1,w_2})}_{bd_{w_1,w_2}}(j)$, the simulated evaluation of $A_{w_1,w_2}P_{bd_{w_1,w_2}}(j)$ in (15), becomes smaller rapidly. For the range of SNR shown in Fig. 5, $A_{w_1,w_2}\tilde{P}^{(\tilde{n}_{w_1,w_2})}_{bd_{w_1,w_2}}(j)$ with $w_1 + w_2 - R_2 \geq 5$ turns out to be negligibly small in the summation. For the range of SNR less than 2.0 dB, however, it is time consuming to obtain reliable estimation of $P_b(j) - P'_b(j)$. In Fig. 5, estimated values are shown only for SNR higher than 2.0 dB.

From Figs. 4 and 5, we see that the effect of incorrect decoding on the bit-error probability, $P_b(j) - P'_b(j)$, is negligibly small compared to $P'_b(j)$.

V. An Improved Concatenated Coding Scheme

The bit-error performance of the proposed concatenated coding scheme can be improved by modifying the way of interleaving after the inner code decoding. In the proposed scheme described in Section II, the interleaving is done as follows. For $1 \leq i \leq N_2$ and $1 \leq j \leq 5$, the jth subsegment of the ith segment decoded by the inner code decoder is stored into the ith row of the jth column of the decoding buffer for outer code decoding. With the modified interleaving, such a subsegment is stored into the ith row of the $((i + j - 2)$ mod 5 + 1)th column of the decoding buffer for outer code decoding. By doing this, the differences of the symbol error probabilities among five subsegments are removed and the bit-error probabilities for the five sections are made uniform.

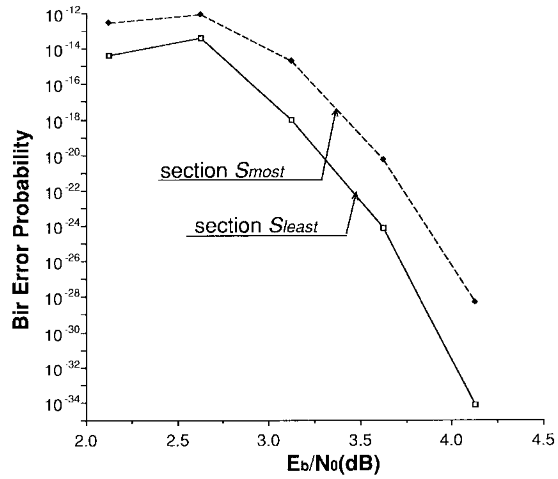

Fig. 5. Simulation results on the $P'_b - P_b$ for sections S_{most} (i.e. $j = 3$) and S_{least} (i.e. $j = 1$).

A. Performance Analysis of the Improved Coding Scheme

The same analysis method presented in the last section can be used to analyze the error performance of the concatenated coding system with the modified interleaving.

Since N_2 is a multiple of five, the approximation $P'_b(j)$ of the bit-error probability is also made uniform by the modified interleaving. Let P'_b denote the approximation of the bit-error probability uniform for the five sections.

P'_b of the improved coding scheme is computed as follows. For $i = 1, \ldots, 5$, let $\bar{K}_i$ be defined as the number of symbols derived from the ith subsegment of the inner code in the information part of the outer code. Similarly, for $i = 1, \ldots, 5$, let $\bar{R}_i$ be defined as the number of symbols derived from the ith subsegment of the inner code in the redundancy part of the outer code. Let a set S of sequences 10-tuples of integers be defined as

$$S \triangleq \Bigg\{(k_1, \ldots, k_5, r_1, \ldots, r_5) : 0 \leq k_i \leq \bar{K}_i \text{ and } 0 \leq r_i \leq \bar{R}_i \text{ for } i = 1, \ldots, 5 \text{ and } \sum_{i=1}^{5}(k_i + r_i) > t_2\Bigg\}. \tag{16}$$

P'_b of the improved coding scheme is computed as

$$P'_b = \sum_{(k_1,\ldots,k_5,r_1,\ldots,r_5)\in S} \Bigg\{\Bigg(\sum_{i=1}^{5} \tilde{w_E}(i)k_i\Bigg) \times \prod_{i=1}^{5}(1 - p(0,i))^{k_i+r_i} p(0,i)^{\bar{K}_i+\bar{R}_i-k_i-r_i}\Bigg\}. \tag{17}$$

Table III shows the average of $P'_b(j)$ over the five sections before and after improving, and the improving ratio, which

TABLE III
COMPARISON OF P'_b BEFORE IMPROVING AND AFTER IMPROVING

Eb/No(dB)	P'_b before improving	P'_b after improving	Improving ratio
2.6	1.94e-02	1.94e-02	1.0
3.1	8.78e-05	4.50e-05	0.5
3.6	2.46e-10	7.16e-11	0.3
4.1	2.04e-18	4.43e-19	0.2

means P'_b (after improving)/P'_b (before improving). From the table, we see that the improved scheme gives lower bit-error probability for most of the practical range of SNR.

The proposed interleaving scheme at the decoding stage can be generalized to any interleaving depth m. In general, for $1 \leq i \leq N_2$ and $1 \leq j \leq m$, the jth subsegment of the ith segment decoded by the inner code decoder is stored into the ith row of the $((i+j-2) \bmod m+1)$th column of the decoding buffer for the outer code decoding.

VI. CONCLUSION

In this paper, we have presented a method for analyzing and evaluating the bit-error performance of a class of concatenated coding systems. This method allows us to evaluate the bit-error probability accurately for these concatenated coding systems at low block error probabilities where conventional simulation methods become infeasible. A specific concatenated coding system was used to illustrate the method. The specific system is being considered for NASA's high-performance and high-speed satellite communications. The trellis structure and Viterbi decoding complexity of the (64, 40, 8) block inner code were presented.

To improve the bit-error performance of the considered concatenated coding systems, a specific interleaving scheme at the decoding stage was presented. This interleaving scheme reduces the difference of bit-error probabilities among the sections.

APPENDIX
SPLIT WEIGHT ENUMERATORS FOR MDS CODES

Let C be an (N, K) MDS code over GF(q). For a codeword $\boldsymbol{v}$ of C, let $s(\boldsymbol{v})$ denote the set of indexes of nonzero components of $\boldsymbol{v}$. Let N_1 and N_2 be nonnegative integers such that $N_1 + N_2 = N$, and let $I_1 \triangleq \{1, 2, \ldots, N_1\}, I_2 \triangleq \{N_1+1, N_1+2, \ldots, N\}$. For $0 \leq w_1 \leq N_1$ and $0 \leq w_2 \leq N_2$, let A_{w_1,w_2} denote the number of codewords $\boldsymbol{v}$'s in C such that

$$|s(\boldsymbol{v}) \cap I_i| = w_i \qquad \text{for } i = 1, 2 \tag{A.1}$$

where, for a finite set X, $|X|$ denotes the coordinality of X. For subsets $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$, let $N(J_1, J_2)$ be the set of codewords $\boldsymbol{v}$'s in C such that

$$s(\boldsymbol{v}) \cap I_i \subseteq J_i \qquad \text{for } i = 1, 2. \tag{A.2}$$

Let $|J_i| = t_i$ with $i = 1, 2$. Then we have [13] that

$$|N(J_1, J_2)| = 1 \qquad \text{for } t_1 + t_2 \leq R \tag{A.3}$$
$$= q^{t_1+t_2-R} \qquad \text{for } t_1 + t_2 > R \tag{A.4}$$

where $R \triangleq N - K$. For $0 \leq t_i \leq N_i$ with $i = 1, 2$, let U_{t_1,t_2} be defined as

$$U_{t_1,t_2} \triangleq \sum_{\substack{J_1 \subseteq I_1 \\ |J_1| = t_1}} \sum_{\substack{J_2 \subseteq I_2 \\ |J_2| = t_2}} |N(J_1, J_2)|. \tag{A.5}$$

It follows from (A.3)–(A.5) that

$$U_{t_1,t_2} = \binom{N_1}{t_1}\binom{N_2}{t_2} \qquad \text{for } t_1 + t_2 \leq R \tag{A.6}$$
$$= \binom{N_1}{t_1}\binom{N_2}{t_2} q^{t_1+t_2-R} \qquad \text{for } t_1 + t_2 > R. \tag{A.7}$$

Since a codeword $\boldsymbol{v}$ in C such that $|s(\boldsymbol{v}) \cap I_i| = w_i$ for $i = 1, 2$ is counted $\binom{N_1 - w_1}{t_1 - w_1}\binom{N_2 - w_2}{t_2 - w_2}$ times in the sum U_{t_1,t_2}, the following equality holds

$$U_{t_1,t_2} = \sum_{w_1=0}^{t_1} \binom{N_1 - w_1}{t_1 - w_1} \sum_{w_2=0}^{t_2} \binom{N_2 - w_2}{t_2 - w_2} A_{w_1,w_2}$$
$$\text{for } 0 \leq t_1 \leq N_1 \text{ and } 0 \leq t_2 \leq N_2. \tag{A.8}$$

Let A'_{w_1,t_2} be defined as

$$A'_{w_1,t_2} \triangleq \sum_{w_2=0}^{t_2} \binom{N_2 - w_2}{t_2 - w_2} A_{w_1,w_2}$$
$$\text{for } 0 \leq w_1 \leq N_1 \text{ and } 0 \leq t_2 \leq N_2. \tag{A.9}$$

From (A.8) and (A.9), we have

$$U_{t_1,t_2} = \sum_{w_1=0}^{t_1} \binom{N_1 - w_1}{t_1 - w_1} A'_{w_1,t_2}$$
$$= \sum_{w_1=0}^{t_1} \binom{N_1 - w_1}{N_1 - t_1} A'_{w_1,t_2} \tag{A.10}$$

for $0 \leq t_1 \leq N_1$ and $0 \leq t_2 \leq N_2$. By using the principle of Inclusion and Exclusion, we have that, for $0 \leq w_1 \leq N_1$ and $0 \leq t_2 \leq N_2$,

$$A'_{w_1,t_2} = \sum_{i=0}^{w_1} (-1)^i \binom{N_1 - w_1 + i}{i} U_{w_1-i,t_2}. \tag{A.11}$$

By applying the principle of Inclusion and Exclusion to (A.9), the following formula is derived

$$A_{w_1,w_2} = \sum_{i=0}^{w_2} (-1)^i \binom{N_2 - w_2 + i}{i} A'_{w_1,w_2-i} \tag{A.12}$$

for $0 \leq w_1 \leq N_1$ and $0 \leq w_2 \leq N_2$. A formula for A_{w_1,w_2} follows from (A.6), (A.7), (A.11), and (A.12). △△

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