Eigenvalue Bounds Versus Semidefinite Relaxations for the Quadratic Assignment Problem

Kurt M. Anstreicher
Dept. of Management Sciences
University of Iowa
Iowa City, IA 52242

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Abstract

It was recently demonstrated that a well-known eigenvalue bound for the Quadratic Assignment Problem (QAP) actually corresponds to a semidefinite programming (SDP) relaxation. However, for this bound to be computationally useful the assignment constraints of the QAP must first be eliminated, and the bound then applied to a lower-dimensional problem. The resulting "projected eigenvalue bound" is one of the best available bounds for the QAP, especially when considering the quality of bounds relative to the complexity of obtaining them. In this paper we show that the projected eigenvalue bound also corresponds to an SDP relaxation of the original QAP.

Keywords: Quadratic Assignment Problem, Eigenvalue Bounds, Semidefinite Programming.

1 Introduction

The Quadratic Assignment Problem (QAP) is a well-studied problem in discrete optimization. For recent surveys see for example [6], [7], and [16]. In this paper we consider the "Koopmans-Beckmann" form of the problem, which can be written

$$\begin{aligned} \text{QAP}(A,B,C): \quad \min \quad & \text{tr}(AXB + C)X^T \\ \text{s.t.} \quad & X \in \Pi, \end{aligned}$$

where A, B and C are $n \times n$ matrices, tr denotes the trace of a matrix, and Π is the set of $n \times n$ permutation matrices. Throughout we assume that A and B are symmetric. We write QAP(A,B) for the "homogenous" problem with $C = 0$. QAP(A,B,C) arises naturally in facility planning, and can also be used to model certain other well-known combinatorial optimization problems, such as graph partitioning, and the travelling salesman problem. The problem is of interest both for its applicability and its difficulty. For the general case, $n = 20$ is approximately the current limit on problem size for which an exact optimal solution can be obtained.

Algorithms that attempt to solve the QAP to optimality must incorporate both primal heuristics that obtain good feasible solutions, and lower-bounding methods, in a branch-and-bound structure. At present the greatest obstacle to obtaining provably optimal solutions for QAP problems is the lack of an efficient lower-bounding method that produces reasonably tight bounds. There are a number of different classes of lower-bounding methods for QAP, including

- The Gilmore-Lawler bound (GLB), and related bounds;
- Bounds based on linear programming (LP) relaxations;
- Eigenvalue-based bounds;
- Bounds based on semidefinite programming (SDP).

Research is active in all four of these areas. In this paper we will concentrate on eigenvalue bounds, and their relationship to SDP. See for example [11], [15], [18], and references therein for recent work on variants of the GLB, and LP relaxations.

Semidefinite programming refers to optimization over matrices that are constrained to be positive semidefinite. Although the potential applicability of SDP has been known for some time, it is only recently that interior-point algorithms have provided a practical solution approach. See [1] and [19] for descriptions of different types of problems that can be formulated as SDPs. SDP-based approaches to the QAP have been considered by [13] and [20]. The bounds for QAP developed in these two papers are highly competitive, but the solution times required on modest-sized problems exceed what could realistically be expended at each node in a branch-and-bound tree.

The basic eigenvalue bound for QAP was introduced in [8], and has been modified in a variety of ways; see for example [10] and [17]. It was recently demonstrated [4] that the simplest eigenvalue bound for QAP(A, B) actually corresponds to a semidefinite relaxation of the problem. This result is potentially interesting because the work to obtain the eigenvalue bound is far less than that required to solve a general SDP. Unfortunately, the basic eigenvalue bound for QAP is known to be too weak to be computationally useful. One technique for strengthening the bound, from [10], is to implicitly enforce the assignment constraints of QAP by first projecting out, or eliminating, these constraints before applying the eigenvalue bound. The resulting "projected eigenvalue bound" is a competitive bound for many problems, especially considering the quality of the bound versus the computational effort required to obtain it.

In the next section we review eigenvalue bounds for QAP, including the projected eigenvalue bound PB(A, B, C). In Section 3 we consider the SDP interpretation of the basic eigenvalue bound proved in [4], and use this interpretation to derive a new semidefinite programming problem, SDP$^+(A, B)$. We prove that PB(A, B, C) corresponds to first applying a simple transformation to QAP, and then using SDP$^+(\cdot, \cdot)$ to bound the quadratic term. (The linear term is bounded separately by solving a linear assignment problem.) We show that the "implicit" semidefinite program SDP$^+(A, B)$ is closely related to the SDP relaxations for QAP proposed in [13] and [20].

Notation. We use $\operatorname{tr} A$ to denote the trace of a square matrix A, and $A \bullet B = \operatorname{tr}(AB^T)$. For symmetric matrices A and B we use $B \succeq A$ to denote that $B - A$ is positive semidefinite, and $B \succ A$ to denote that $B - A$ is positive definite. We use e to denote a vector of arbitrary dimension with each component equal to one, and $E = ee^T$. The Kronecker product of matrices A and B is denoted $A \otimes B$. See [9] or [12] for basic properties of Kronecker products. We sometimes abuse notation by writing, for example, $e = (e \otimes e)$, where e on the left side is a vector in $\Re^{n^2}$, and each e on the right is a vector in $\Re^n$. For an $n \times n$ symmetric matrix A, $\lambda(A) \in \Re^n$ denotes the vector of eigenvalues of A. For a vector x, $\operatorname{Diag}(x)$ is the diagonal matrix with diagonal entries equal to the components of x, and for a square matrix X, $\operatorname{diag}(X)$ is the vector whose components are the diagonal entries of X.

We use $\mathcal{O}$ to denote the set of orthogonal matrices ($XX^T = XX^T = I$), $\mathcal{E}$ to denote the set of doubly-stochastic matrices ($Xe = X^Te = e$), and Π to denote the set of permutation matrices. The "minimal product" of two vectors x and y in $\Re^n$ is denoted $\langle x, y\rangle_-$, and is defined by

$$\langle x, y\rangle_- = \min_{\pi} \prod_{i=1}^{n} x_i y_{\pi(i)},$$

where $\pi(\cdot)$ is a permutation of $1, 2, \ldots, n$. It is easy to show that if $x_1 \leq x_2 \leq \ldots \leq x_n$, and $y_1 \geq y_2 \geq \ldots \geq y_n$, then $\langle x, y\rangle_- = x^T y$. We sometimes use $r(A)$ to denote the vector of row sums of a matrix A, $r(A) = Ae$, and $s(A)$ to denote the sum of the entries of A, $s(A) = e^T Ae = e^T r(A)$.

Throughout the paper we use the convention of letting the name of an optimization problem, such as $\mathrm{QAP}(A, B, C)$, also refer to the solution value of the problem.

2 Eigenvalue Bounds for QAP

One well-known relaxation for QAP is based on relaxing $X \in \Pi$ to $X \in \mathcal{O}$, and further separating the linear and quadratic terms in the objective. The result is:

$$\min_{X \in \mathcal{O}} \quad \operatorname{tr} AXBX^T + \mathrm{LAP}(C), \tag{1}$$

where LAP(C) is a linear assignment problem with cost matrix C. The relaxation (1) is potentially useful because a closed-form expression exists for the quadratic term. Specifically, it is known [8, 17] that

$$\min_{X\in\mathcal{O}} \quad \mathrm{tr}\, AXBX^T = \langle\lambda(A), \lambda(B)\rangle_-, \tag{2}$$

and therefore (1) can be computed by performing spectral decompositions of A and B, and solving LAP(C). Unfortunately, however, the basic eigenvalue bound from (1) is generally a very weak bound for QAP, even when $C = 0$.

There are several ways to improve (1). One approach is based on first performing perturbations on A, B and C that leave the objective invariant for $X \in \Pi$, and then evaluating the bound from (1) using the perturbed data. Specifically, let

$$A' = A + eg^T + ge^T + \mathrm{Diag}(r), \tag{3a}$$

$$B' = B + eh^T + he^T + \mathrm{Diag}(s), \tag{3b}$$

$$C' = C - 2[Aeh^T + ge^T B + gs^T + rh^T + ngh^T + (e^T g)eh^T] - [as^T + rb^T + rs^T]. \tag{3c}$$

where $a = \mathrm{diag}(A)$, $b = \mathrm{diag}(B)$ and g, h, r, and s are all in $\Re^n$. It is then easy to show that QAP(A', B', C') = QAP(A, B, C). (**Note:** Formulas similar to those in (3) appear with sign errors in a number of standard references on QAP, including [10] and [16]). One choice for the perturbation vectors g, h, r, and s, described in [8, 17] is based on minimizing the *spectral variance* of A' and B'. Another possibility [17], which we will refer to as the *parametric eigenvalue bound*, is obtained by approximately maximizing the eigenvalue bound

$$\langle\lambda(A'), \lambda(B')\rangle_- + \mathrm{LAP}(C')$$

over the perturbation vectors g, h, r, and s. The result is one of the strongest known bounds for the QAP, but performing the approximate maximization is quite difficult due to the fact that as a function of the perturbations the bound is a nondifferentiable, nonconcave function.

A different approach to improving the basic eigenvalue bound (1) was introduced in [10]. The idea of this improvement is to continue to work with an orthogonal relaxation of the quadratic term, but to enforce the assignment constraints $X \in \mathcal{E}$ that are ignored in (1). The mechanism to do this is provided by the following result.

Proposition 2.1 [10, Lemma 3.1] *Let X be an $n \times n$ matrix with $X \in \mathcal{O} \cap \mathcal{E}$. Then there is an $(n-1) \times (n-1)$ orthogonal matrix $\hat{X}$ such that $X = V\hat{X}V^T + (1/n)ee^T$, where V is an $n \times (n-1)$ matrix whose columns are an orthonormal basis for the nullspace of e^T. Conversely, if $\hat{X}$ is an $(n-1) \times (n-1)$ orthogonal matrix, then $X = V\hat{X}V^T + (1/n)ee^T \in \mathcal{O} \cap \mathcal{E}$.*

In [10], Proposition 2.1 is used to obtain the *projected eigenvalue bound* $\mathrm{PB}(A, B, C)$ for QAP described in the following theorem.

Theorem 2.2 *Let V be an $n \times (n-1)$ matrix whose columns are an orthonormal basis for the nullspace of e^T, and define $\hat{A} = V^T AV$, $\hat{B} = V^T BV$, $D = C + (2/n)r(A)r(B)^T$. Let* $\mathrm{PB}(A, B, C) = \langle \lambda(\hat{A}), \lambda(\hat{B}) \rangle_- + \mathrm{LAP}(D) - s(A)s(B)/n^2$. *Then*

1. $\mathrm{QAP}(A, B, C) \geq \mathrm{PB}(A, B, C)$
2. $\mathrm{PB}(A, B) = \langle \lambda(\hat{A}), \lambda(\hat{B}) \rangle_- + (2/n)\langle r(A), r(B) \rangle_- - s(A)s(B)/n^2$.
3. *If e is an eigenvector of either A or B,* $\mathrm{PB}(A, B) = \langle \lambda(\hat{A}), \lambda(\hat{B}) \rangle_- + s(A)s(B)/n^2$.

Proof: Part 1 is [10, Theorem 4.1]. Parts 2 and 3 are proved in Corollaries 4.1 and 4.2 of [10], but we describe the arguments here also. In the case that $C = 0$ we have $D = (2/n)r(A)r(B)^T$, and it is then easy to show that $\mathrm{LAP}(D) = (2/n)\langle r(A), r(B) \rangle_-$, which gives 2. To show 3, assume that $Ae = r(A) = \mu e$, from which it follows that $\mu = s(A)/n$. Then $\langle r(A), r(B) \rangle_- = \mu e^T r(B) = s(A)s(B)/n$, and 2 implies 3. The argument when e is an eigenvector of B is similar. □

The projected eigenvalue bound of Theorem 2.2 is one of the best available bounds for the general QAP, especially considering the quality of the bound versus the effort required to compute it. Computational results reported in [10] show that the projected eigenvalue bound is often close to the parametric eigenvalue bound, but is much more practical to compute.

3 Semidefinite Programming Relaxations

A new interpretation of the basic eigenvalue bound (1) in terms of semidefinite programming was recently given in [4]. For an $n^2 \times n^2$ matrix Y, let $Y_{[ij]}$ denote the $n \times n$ matrix which is the ij "block" of Y, $i, j = 1, \ldots, n$. In other words,

$$Y = \begin{pmatrix} Y_{[11]} & \ldots & Y_{[1n]} \\ \vdots & \ddots & \vdots \\ Y_{[n1]} & \ldots & Y_{[nn]} \end{pmatrix}.$$

Define [20] the linear operators from $\Re^{n^2 \times n^2}$ to $\Re^{n \times n}$:

$$\begin{aligned} \text{bdiag}(Y) &= \sum_{i=1}^{n} Y_{[ii]} \\ (\text{odiag}(Y))_{ij} &= \text{tr}\, Y_{[ij]}, \quad i, j = 1, \ldots, n. \end{aligned}$$

It is then easy to show that bdiag$(\cdot)$ and odiag$(\cdot)$ are the adjoints of the operators $S \to I \otimes S$ and $T \to T \otimes I$, from $\Re^{n \times n}$ to $\Re^{n^2 \times n^2}$, respectively. Consider the following pair of semidefinite programming problems:

$$\begin{array}{llll} \text{SDP}(A,B): & \min & (B \otimes A) \bullet Y \\ & \text{s.t.} & \text{bdiag}(Y) = I \\ & & \text{odiag}(Y) = I \\ & & Y \succeq 0, \end{array} \qquad \begin{array}{lll} \text{SDD}(A,B): & \max & \text{tr}\, S + \text{tr}\, T \\ & \text{s.t.} & (I \otimes S) + (T \otimes I) \preceq (B \otimes A) \\ & & S = S^T,\ T = T^T. \end{array}$$

Theorem 3.1 $\text{SDP}(A,B) = \text{SDD}(A,B) = \langle \lambda(A), \lambda(B) \rangle_-$.

Proof: $\text{SDD}(A,B) = \langle \lambda(A), \lambda(B) \rangle_-$ is proved in [4, Theorem 3.2]. $\text{SDP}(A,B) = \text{SDD}(A,B)$ follows from the fact that these are dual semidefinite programming problems, both of which have interior solutions; see for example [14, Theorem 4.2.1]. □

The problem $\text{SDP}(A,B)$ can be viewed as a semidefinite relaxation of $\text{QAP}(A,B)$. Note that $\text{tr}\, AXBX^T = \mathbf{vec}(X)^T (B \otimes A) \mathbf{vec}(X) = (B \otimes A) \bullet \mathbf{vec}(X) \mathbf{vec}(X)^T$, and clearly $\mathbf{vec}(X) \mathbf{vec}(X)^T \succeq 0$. The equality constraints of $\text{SDP}(A,B)$ are relaxations of the orthogonality condition on the matrix X of QAP. Specifically, if $Y = \mathbf{vec}(X) \mathbf{vec}(X)^T$, then $Y_{[ij]} = X_i X_j^T$, where X_i is the ith column of X. It follows that for such a Y,

$$\text{bdiag}(Y) = XX^T, \qquad \text{odiag}(Y) = X^T X.$$

The fact that $\langle \lambda(A), \lambda(B) \rangle_- = \text{SDD}(A,B)$ can be viewed as a rather surprising Lagrangian strong duality result for a nonconvex problem [4]. It is particularly interesting that this result holds only with *both* of the conditions $\text{bdiag}(Y) = I$, $\text{odiag}(Y) = I$ enforced in $\text{SDP}(A, B)$, despite the fact that the original constraints $XX^T = I$ and $X^T X = I$ are completely equivalent. See [3] for an analog of Theorem 3.1 for a relaxation of $\text{QAP}(A, B)$ with the semidefinite inequality $XX^T \preceq I$ in place of the orthogonality condition $XX^T = I$.

Theorem 3.1 shows that the original eigenvalue bound from (1) corresponds to using the semidefinite programming relaxation $\text{SDP}(A, B)$ in place of the quadratic term of QAP. Unfortunately, as noted above, it is known that this bound is in general too weak to be of use computationally. However, it is obvious that Theorem 3.1 can also be applied to obtain an SDP representation for the term $\langle \lambda(\hat{A}), \lambda(\hat{B}) \rangle_-$ in the projected eigenvalue bound of Theorem 2.2. Our goal now is to show that this SDP representation of the "projected" problem can be lifted back to the original problem to obtain a new, stronger SDP relaxation of the original QAP.

From Theorem 3.1, $\langle \lambda(\hat{A}), \lambda(\hat{B}) \rangle_- = \text{SDD}(\hat{A}, \hat{B})$, which can be written as

$$\begin{array}{ll} \max & \operatorname{tr} \hat{S} + \operatorname{tr} \hat{T} \\ \text{s.t.} & (I \otimes \hat{S}) + (\hat{T} \otimes I) \preceq \hat{B} \otimes \hat{A}, \end{array}$$

where $\hat{A} = V^T AV$, $\hat{B} = V^T BV$. (Henceforth we consider the symmetry constraints on $\hat{S}$ and $\hat{T}$ to be implicit.) But any $\hat{S}$ and $\hat{T}$ can be written in the form $\hat{S} = V^T SV$, $\hat{T} = V^T TV$, for $n \times n$ symmetric matrices S and T. Since $V^T V = I$, $\text{SDD}(\hat{A}, \hat{B})$ is then equivalent to the problem

$$\begin{array}{ll} \max & \operatorname{tr} V^T SV + \operatorname{tr} V^T TV \\ \text{s.t.} & (V^T \otimes V^T)\Big[(B \otimes A) - (I \otimes S) - (T \otimes I)\Big](V \otimes V) \succeq 0. \end{array} \tag{4}$$

The following proposition is well known from the theory of augmented Lagrangian methods; see for example [2, Corollary 12.9].

Proposition 3.2 *Let H be a $k \times k$ symmetric matrix, and let F be an $m \times k$ matrix. Let Z be a matrix whose columns are a basis for the nullspace of F. Then the following three conditions are equivalent:*

1. $x^T Hx > 0$ *for all* $x \neq 0$ *having* $Fx = 0$.

2. $Z^T HZ \succ 0$.

3. $H + \rho F^T F \succ 0$ *for all sufficiently large* ρ.

In our application, it would be convenient if a "semidefinite" version of Proposition 3.2, with "$\geq$" replacing "$>$" in part 1, and "$\succeq$" replacing "$\succ$" in parts 2 and 3, were true. Unfortunately this is not the case; see [5].

Let F be the $2n \times n^2$ matrix

$$F = \begin{pmatrix} e^T \otimes I \\ I \otimes e^T \end{pmatrix}.$$

The matrix F arises naturally in the representation of the assignment constraints of QAP when the matrix X is written as a vector $\mathbf{vec}(X)$. Specifically, the constraints $Xe = e$, $X^T e = e$ are exactly equivalent to $F\,\mathbf{vec}(X) = e$.

Lemma 3.3 *The columns of* $V \otimes V$ *are a basis for the nullspace of* F.

Proof: The columns of $V \otimes V$ are certainly in the nullspace of F, since $(e^T \otimes I)(V \otimes V) = e^T V \otimes V = 0$, and $(I \otimes e^T)(V \otimes V) = V \otimes e^T V = 0$. Moreover it follows from the fact that $(V, e) \otimes (V, e)$ is a nonsingular matrix that the columns of $V \otimes V$ are independent. Finally, it is very well known that the rank of F is $2n - 1$, and therefore the dimension of the nullspace of F is $n^2 - (2n - 1) = (n - 1)^2$, which is exactly the number of columns of $V \otimes V$. $\square$

Motivated by Proposition 3.2 and Lemma 3.3, we define the following semidefinite program.

$$\widehat{\mathrm{SDD}}(A, B): \qquad \begin{array}{ll} \sup & VV^T \bullet S + VV^T \bullet T \\ \text{s.t.} & (I \otimes S) + (T \otimes I) - \rho F^T F \preceq B \otimes A. \end{array}$$

In the next lemma we demonstrate that $\mathrm{SDD}(\hat{A}, \hat{B})$ and $\widehat{\mathrm{SDD}}(A, B)$ are equivalent.

Lemma 3.4 *If* S, T, ρ *are feasible in* $\widehat{\mathrm{SDD}}(A, B)$, *then* $\hat{S} = V^T SV$, $\hat{T} = V^T TV$ *are feasible in* $\mathrm{SDD}(\hat{A}, \hat{B})$, *and* $\mathrm{tr}\,\hat{S} + \mathrm{tr}\,\hat{T} = VV^T \bullet S + VV^T \bullet T$. *Conversely if* $\hat{S}$ *and* $\hat{T}$ *are feasible in* $\mathrm{SDD}(\hat{A}, \hat{B})$, *then for every* $\epsilon > 0$ *there are* $S_\epsilon, T_\epsilon, \rho_\epsilon$ *feasible in* $\widehat{\mathrm{SDD}}(A, B)$ *such that* $VV^T \bullet S_\epsilon + VV^T \bullet T_\epsilon = \mathrm{tr}\,\hat{S} + \mathrm{tr}\,\hat{T} - \epsilon$.

Proof: Assume that S, T, ρ are feasible in $\widehat{\mathrm{SDD}}(A,B)$, and let $H = B\otimes A - (I\otimes S) - (T\otimes I)$. If $x \in \Re^{n^2}$ is in the nullspace of F, then $x^T H x = x^T(H+\rho F^T F)x \geq 0$, since $H + \rho F^T F \succeq 0$. Using Lemma 3.3, it follows that $(V^T \otimes V^T)H(V\otimes V) \succeq 0$, so S and T are feasible in (4). Therefore $\hat{S}$ and $\hat{T}$ are feasible in $\mathrm{SDD}(\hat{A},\hat{B})$, and $\mathrm{tr}\, \hat{S} = VV^T \bullet S$, $\mathrm{tr}\, \hat{T} = VV^T \bullet T$.

Next assume that $\hat{S}, \hat{T}$ are feasible in $\mathrm{SDD}(\hat{A},\hat{B})$. Then $S = V\hat{S}V^T$, $T = V\hat{T}V^T$ are feasible in (4). For $\epsilon > 0$ let $S_\epsilon = S - [\epsilon/(n-1)]I$, and $H_\epsilon = B\otimes A - (I\otimes S_\epsilon) - (T\otimes I)$. Then $\mathrm{tr}\, V^T S_\epsilon V + \mathrm{tr}\, V^T T V = \mathrm{tr}\, \hat{S} + \mathrm{tr}\, \hat{T} - \epsilon$, and $(V^T\otimes V^T)H_\epsilon(V\otimes V) \succ 0$. Applying Proposition 3.2, and Lemma 3.3, there is a ρ_ϵ so that $H_\epsilon + \rho_\epsilon F^T F \succ 0$, and therefore $S_\epsilon, T, \rho_\epsilon$ is feasible in $\widehat{\mathrm{SDD}}(A,B)$. □

The dual of $\widehat{\mathrm{SDD}}(A,B)$ is the semidefinite program

$$
\begin{array}{lll}
\widehat{\mathrm{SDP}}(A,B): & \min & (B\otimes A)\bullet \bar{Y} \\
 & \text{s.t.} & \mathrm{bdiag}(\bar{Y}) = VV^T \\
 & & \mathrm{odiag}(\bar{Y}) = VV^T \\
 & & \bar{Y} \bullet F^T F = 0 \\
 & & \bar{Y} \succeq 0.
\end{array}
$$

The fact that $\widehat{\mathrm{SDD}}(A,B)$ has an interior and a bounded objective implies that $\widehat{\mathrm{SDP}}(A,B) = \widehat{\mathrm{SDD}}(A,B)$, and the solution value in $\widehat{\mathrm{SDP}}(A,B)$ is attained; see [14, Theorem 4.2.1]. From Theorem 3.1 and Lemma 3.4 we then have $\widehat{\mathrm{SDP}}(A,B) = \langle \lambda(\hat{A}), \lambda(\hat{B})\rangle_-$.

The problem $\widehat{\mathrm{SDP}}(A,B)$ corresponds to an SDP relaxation of $\mathrm{QAP}(A,B)$, but this can be seen more clearly by reformulating $\widehat{\mathrm{SDP}}(A,B)$ in terms of the matrix $Y = \bar{Y} + (1/n^2)E$. It is straightforward to compute that for an $n^2\times n^2$ matrix E,

$$
\mathrm{bdiag}(E) = nE \tag{5a}
$$

$$
\mathrm{odiag}(E) = nE \tag{5b}
$$

$$
E \bullet F^T F = 2n^3. \tag{5c}
$$

In addition, note that $(V,v)(V,v)^T = I$, where $v = (1/\sqrt{n})e$, and therefore $VV^T = I - vv^T = I - (1/n)E$. It follows that $\bar{Y}$ being feasible in $\widehat{\mathrm{SDP}}(A,B)$ is equivalent to $Y = \bar{Y} + (1/n^2)E$

being feasible in the semidefinite programming problem

$$\begin{array}{llll} \mathrm{SDP}^+(A,B): & \min & (B\otimes A)\bullet Y \\ & \text{s.t.} & \mathrm{bdiag}(Y) = I \\ & & \mathrm{odiag}(Y) = I \\ & & Y\bullet F^TF = 2n \\ & & Y \succeq (1/n^2)E. \end{array}$$

Finally,

$$\begin{aligned} (B\otimes A)\bullet E &= \mathrm{tr}(B\otimes A)(ee^T\otimes ee^T) \\ &= \mathrm{tr}(B\otimes A)(e\otimes e)(e^T\otimes e^T) \\ &= \mathrm{tr}(e^T\otimes e^T)(B\otimes A)(e\otimes e) \\ &= s(A)s(B), \end{aligned} \tag{6}$$

so $Y = \bar{Y} + (1/n^2)E$ implies that

$$\mathrm{SDP}^+(A,B) = \widehat{\mathrm{SDP}}(A,B) + \frac{s(A)s(B)}{n^2} = \langle\lambda(\hat{A}),\lambda(\hat{B})\rangle_- + \frac{s(A)s(B)}{n^2}. \tag{7}$$

The problem $\mathrm{SDP}^+(A,B)$ is clearly a stronger semidefinite relaxation of $\mathrm{QAP}(A,B)$ than $\mathrm{SDP}(A,B)$. The additional equality constraint $Y\bullet F^TF = 2n$ corresponds to a relaxation of the assignment constraints $Xe = X^Te = e$ of QAP. Note that $\mathrm{tr}\,YF^TF = \mathrm{tr}\,FYF^T$, so the constraint $\bar{Y}\bullet F^TF = 0$ of $\widehat{\mathrm{SDP}}(A,B)$ is equivalent to $\mathrm{tr}\,F\bar{Y}F^T = 0$. However $F\bar{Y}F^T \succeq 0$, and therefore $\mathrm{tr}\,F\bar{Y}F^T = 0$ is equivalent to $F\bar{Y}F^T = 0$. For $Y = \bar{Y} + (1/n^2E)$, the latter is equivalent to $FYF^T = E$. Finally, if $Y = \mathbf{vec}(X)\,\mathbf{vec}(X)^T$, then

$$\begin{aligned} FYF^T &= \begin{pmatrix} e^T\otimes I \\ I\otimes e^T \end{pmatrix} \mathbf{vec}(X)\,\mathbf{vec}(X)^T \begin{pmatrix} e^T\otimes I \\ I\otimes e^T \end{pmatrix}^T \\ &= \begin{pmatrix} \mathbf{vec}(Xe) \\ \mathbf{vec}(e^TX) \end{pmatrix} \begin{pmatrix} \mathbf{vec}(Xe) \\ \mathbf{vec}(e^TX) \end{pmatrix}^T \\ &= \begin{pmatrix} Xe \\ X^Te \end{pmatrix} \begin{pmatrix} Xe \\ X^Te \end{pmatrix}^T, \end{aligned}$$

so $Xe = X^T e = e$ implies that $FYF^T = E$.

Comparing (7) with part 3 of Theorem 2.2, it is clear that $\mathrm{SDP}^+(A,B) = \mathrm{PB}(A,B)$ when e is an eigenvector of either A or B. We next show that in all cases $\mathrm{PB}(A,B,C)$, as defined in Theorem 2.2, corresponds to applying $\mathrm{SDP}^+(\cdot,\cdot)$ to bound the quadratic term of $\mathrm{QAP}(A,B,C)$, after a preliminary transformation (3) that makes e an eigenvalue of A.

Lemma 3.5 *Let $A' = A + eg^T + ge^T$, where $g = (-1/n)Ae$, and let $C' = C + (2/n)Aee^T B = C + (2/n)r(A)r(B)^T$. Then $\mathrm{PB}(A,B,C) = \mathrm{SDP}^+(A',B) + \mathrm{LAP}(C')$.*

Proof: Note that

$$A'e = Ae + (g^T e)e + (e^T e)g = -\frac{s(A)}{n}e,$$

so e is an eigenvector of A', and $s(A') = -s(A)$. From (7), and the fact that $V^T A' V = V^T AV = \hat{A}$, we then have

$$\mathrm{SDP}^+(A',B) = \langle \lambda(\hat{A}), \lambda(\hat{B})\rangle_- + \frac{s(A')s(B)}{n^2} = \langle \lambda(\hat{A}), \lambda(\hat{B})\rangle_- - \frac{s(A)s(B)}{n^2},$$

and $\mathrm{PB}(A,B,C) = \mathrm{SDP}^+(A',B) + \mathrm{LAP}(C')$ follows from the defininition of $\mathrm{PB}(A,B,C)$. □

Lemma 3.5 uses a transformation of A that makes e an eigenvalue of A', but it is easy to see that a similar result holds if an analogous transformation is applied to B instead. As mentioned above, the problem $\mathrm{SDP}^+(A,B)$ can be interpreted as a strengthened semidefinite relaxation of $\mathrm{QAP}(A,B)$. In particular, note that if $X \in \mathcal{O} \cap \mathcal{E}$, then $Y = \mathbf{vec}(X)\,\mathbf{vec}(X)^T$ is feasible for all of the equality constraints in $\mathrm{SDP}^+(A,B)$, and also has $(B \otimes A) \bullet Y = \mathbf{vec}(X)^T (B \otimes A)\,\mathbf{vec}(X) = \mathrm{tr}\, AXBX^T$. However, such a Y will *not* in general be feasible for the constraint $Y \succeq (1/n^2)E$. In the next theorem we show that when e is an eigenvector of A or B, $\mathrm{SDP}^+(A,B)$ is in fact a valid relaxation of $\mathrm{QAP}(A,B)$.

Theorem 3.6 *Assume that $X \in \mathcal{O} \cap \mathcal{E}$, and let $Y = \mathbf{vec}(X)\,\mathbf{vec}(X)^T$, $Y' = Y - (1/n^2)[YE + EY] + (2/n^2)E$. Then Y' is feasible in $\mathrm{SDP}^+(A,B)$. Moreover, if e is an eigenvector of A or B, then $(B \otimes A) \bullet Y' = (B \otimes A) \bullet Y$.*

Proof: To begin, we will show that

$$\begin{aligned} \text{bdiag}(YE) &= \text{bdiag}(EY) = nE \quad &(8a) \\ \text{odiag}(YE) &= \text{odiag}(EY) = nE \quad &(8b) \\ (YE) \bullet F^T F &= (EY) \bullet F^T F = 2n^3. \quad &(8c) \end{aligned}$$

By inspection, using the fact that $Xe = X^T e = e$, we have

$$(YE)_{[ij]} = nX_i e^T, \qquad (EY)_{[ij]} = neX_j^T,$$

for $i, j = 1, \ldots, n$, where X_i is the ith column of X. Then

$$\text{bdiag}(YE) = \sum_{i=1}^{n} (YE)_{ii} = n(Xe)e^T = nE.$$

The argument that $\text{bdiag}(EY) = nE$ is similar, proving (8a). Next,

$$\text{tr}(YE)_{[ij]} = n \,\text{tr}\, X_i e^T = ne^T X_i = n,$$

for all i, j, so $\text{odiag}(YE) = nE$. The argument that $\text{odiag}(EY) = nE$ is similar, proving (8b). Finally $YE = \mathbf{vec}(X)\,\mathbf{vec}(X)^T ee^T = n\,\mathbf{vec}(X)(e^T \otimes e^T)$, and $F^T F = (E \otimes I) + (I \otimes E)$, so

$$(YE) \bullet F^T F = 2n^2 \,\text{tr}\left(\mathbf{vec}(X)(e^T \otimes e^T)\right) = 2n^2 e^T \,\mathbf{vec}(X) = 2n^3,$$

and $(EY) \bullet F^T F = (YE) \bullet F^T F$, proving (8c). Since Y is feasible for the equality constraints in $\text{SDP}^+(A, B)$, (5) and (8) together imply that Y' is also feasible for the equality constraints in $\text{SDP}^+(A, B)$.

Since $X \in \mathcal{O} \cap \mathcal{E}$, Proposition 2.1 implies that there is an $(n-1) \times (n-1)$ orthogonal matrix $\hat{X}$ such that $X = V\hat{X}V^T + (1/n)E$, and therefore

$$\mathbf{vec}(X) = (V \otimes V)\,\mathbf{vec}(\hat{X}) + \frac{1}{n} e. \tag{9}$$

It follows that $Y = \mathbf{vec}(X)\,\mathbf{vec}(X)^T$ can be represented in the form

$$\begin{aligned} Y \;=\; & (V \otimes V)\,\mathbf{vec}(\hat{X})\,\mathbf{vec}(\hat{X})^T (V^T \otimes V^T) + \frac{1}{n} e\,\mathbf{vec}(\hat{X})^T (V^T \otimes V^T) \\ & + \frac{1}{n}(V \otimes V)\,\mathbf{vec}(\hat{X}) e^T + \frac{1}{n^2} ee^T. \end{aligned} \tag{10}$$

Using (9), and writing $\hat{Y} = \mathbf{vec}(\hat{X})\,\mathbf{vec}(\hat{X})^T$, (10) implies that

$$\begin{aligned} Y &= (V \otimes V)\hat{Y}(V^T \otimes V^T) + \frac{1}{n}\left[\mathbf{vec}(X)e^T + e\,\mathbf{vec}(X)^T\right] - \frac{1}{n^2}ee^T \\ &= (V \otimes V)\hat{Y}(V^T \otimes V^T) + \frac{1}{n^2}\left[YE + EY\right] - \frac{1}{n^2}E. \end{aligned}$$

It follows that

$$Y' = Y - \frac{1}{n^2}[YE + EY] + \frac{2}{n^2}E = (V \otimes V)\hat{Y}(V^T \otimes V^T) + \frac{1}{n^2}E \succeq \frac{1}{n^2}E,$$

so Y' is feasible for the semidefinite inequality constraint of $\mathrm{SDP}^+(A, B)$ as well.

Assume that e is an eigenvector of A. Then $Ae = (s(A)/n)e$, and

$$\begin{aligned} (B \otimes A)EY &= (B \otimes A)(e \otimes e)e^T Y \\ &= n(Be \otimes Ae)\,\mathbf{vec}(X)^T \\ &= s(A)(Be \otimes e)\,\mathbf{vec}(X)^T \\ &= s(A)(B \otimes I)(e \otimes e)\,\mathbf{vec}(X)^T. \end{aligned}$$

It follows immediately that

$$\begin{aligned} \mathrm{tr}(B \otimes A)EY &= s(A)(e^T \otimes e^T)(B \otimes I)\,\mathbf{vec}(X) \\ &= s(A)(e^T \otimes e^T)\,\mathbf{vec}(XB) \\ &= s(A)(e^T XBe) \\ &= s(A)s(B). \end{aligned} \tag{11}$$

Combining (6), (11), and the fact that $(B \otimes A) \bullet EY = (B \otimes A) \bullet YE$, we obtain $(B \otimes A) \bullet Y = (B \otimes A) \bullet Y'$. The proof when e is an eigenvector of B is similar. $\square$

It is quite interesting to compare the semidefinite relaxation $\mathrm{SDP}^+(A, B)$ with the SDP relaxations of QAP devised in [13] and [20]. The basic relaxation of [13] is

$$\begin{array}{ll} \min & (B \otimes A) \bullet Y \\ \text{s.t.} & Y \bullet F^T F = 2n \\ & Y \bullet E = n^2 \\ & Y - \mathrm{diag}(Y)\,\mathrm{diag}(Y)^T \succeq 0. \end{array} \tag{12}$$

It is well known that the nonlinear semidefinite inequality $Y - \mathrm{diag}(Y)\,\mathrm{diag}(Y)^T \succeq 0$ can be expressed as a linear semidefinite constraint, for example by writing $\mathcal{A}(Y) \succeq -e_0 e_0^T$, where

$$\mathcal{A}(Y) = \begin{pmatrix} 0 & \mathrm{diag}(Y)^T \\ \mathrm{diag}(Y) & Y \end{pmatrix},$$

and e_0 is the unit vector in $\Re^{n^2+1}$ with a one in the "zero'th" position. Note that (12) does not impose the constraints $\mathrm{bdiag}(Y) = I$, $\mathrm{odiag}(Y) = I$ of $\mathrm{SDP}^+(A, B)$. In addition, it is easy to see that the constraint $\bar{Y} \bullet F^T F = 0$ of $\widehat{\mathrm{SDP}}(A, B)$ implies that $e^T F \bar{Y} F^T e = 0$ (see the discussion below (7)), which is equivalent to $\bar{Y} \bullet E = 0$. As a result the constraint $\bar{Y} \bullet E = n^2$ would be redundant in $\mathrm{SDP}^+(A, B)$. These observations suggest that the bound from (12) could be inferior to $\mathrm{PB}(A, B)$ in some cases. In fact this is true, as demonstrated by comparing the bounds in Table 1 of [13] with the projected eigenvalue bounds for the same problems (see for example Table 3 of [20]).

For a homogenous problem ($C = 0$) the basic SDP bound of [20] is very similar to $\mathrm{SDP}^+(A, B)$, the main difference being the representation of the assignment constraints. (The general construction of [20] also provides an SDP bound for problems with $C \neq 0$, but we omit the details here.) Since the original assignment constraints of QAP can be written $F\,\mathbf{vec}(X) - e = 0$, where $e \in \Re^{2n}$, it is certainly true that

$$\mathbf{vec}(X)^T F^T F\,\mathbf{vec}(X) - 2e^T F\,\mathbf{vec}(X) + 2n = 0. \tag{13}$$

For $X \in \Pi$ and $Y = \mathbf{vec}(X)\,\mathbf{vec}(X)^T$ we have $\mathbf{vec}(X) = \mathrm{diag}(Y)$, and $e^T F = 2e^T$, so (13) can be written

$$Y \bullet F^T F - 4e^T\,\mathrm{diag}(Y) + 2n = 0.$$

The basic SDP relaxation for $\mathrm{QAP}(A, B)$ from [20] is

$$\begin{array}{ll} \min & (B \otimes A) \bullet Y \\ \text{s.t.} & \mathrm{bdiag}(Y) = I \\ & \mathrm{odiag}(Y) = I \\ & Y \bullet F^T F - 4e^T\,\mathrm{diag}(Y) = -2n \\ & Y - \mathrm{diag}(Y)\,\mathrm{diag}(Y)^T \succeq 0. \end{array} \tag{14}$$

Note, however, that $\text{bdiag}(Y) = I$ implies that $e^T \text{diag}(Y) = n$, so the constraint $Y \bullet F^T F - 4e^T \text{diag}(Y) = -2n$ of (14) is actually equivalent to $Y \bullet F^T F = 2n$. In the computational results reported in [20], the bound from (14) is never worse than $\text{PB}(A, B)$. However, it is interesting to note that there are several problems for which these two bounds coincide (see tables 1 and 3 of [20]).

Both [13] and [20] consider strengthenings of the basic SDP bounds in (12) and (14), respectively. These improved bounds are based on two classes of constraints that are valid for $Y = \mathbf{vec}(X)\,\mathbf{vec}(X)^T$, $X \in \Pi$;

- all components of Y should be nonnegative,
- certain components of Y should be zero.

By imposing additional constraints of one or both of the above types, [13] and [20] obtain substantial improvements over the basic SDP bounds from (12) and (14). Unfortunately, however, the computational cost of obtaining these improved bounds is considerable.

4 Conclusion

We have shown that the well-known projected eigenvalue bound for QAP corresponds to first applying a simple transformation to the problem, and then using a semidefinite relaxation to bound the quadratic term. The implicit semidefinite relaxation is closely related to SDP relaxations for QAP proposed in [13] and [20]. Besides its purely theoretical interest, there are several possible applications for this result. For example, because the projected eigenvalue bound corresponds to a particular $X \in \mathcal{O} \cap \mathcal{E}$, this "solution" might be useful for warm-starting a stronger SDP relaxation of QAP. In addition, the knowlege that the bound corresponds to implicitly solving a convex optimization problem may make it possible to derive stronger bounds that do not require explicit solution of an SDP.

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