Correspondence

A State-Space Approach to Analysis of Almost Periodic Nonlinear Systems with Sector Nonlinearities

Seung Jean Kim and In-Joong Ha

***Abstract*—The nonlinear system considered in this paper consists of two components—a linear system and a time-varying nonlinear element in the feedback connection—and is almost periodic. It is shown using a state-space approach (or Lyapunov approach) that if the well-known circle criterion is satisfied, then there exists an almost periodic solution which is uniformly asymptotically stable in the large. Furthermore, it is also shown that the spectra of the almost periodic solution can be characterized by those of the input and the time-varying nonlinearity.**

***Index Terms*— Almost periodic, circle criterion, nonlinear feedback system.**

I. Introduction

The nonlinear feedback system considered in this paper consists of a linear system and a nonlinear element connected in the feedback configuration as shown in Figs. 1 and 2. During the last four decades, there has been a lot of research on the stability of solutions of nonlinear feedback systems of the forms in Figs. 1 and 2 [1]–[6]. While most of the prior works discussed the system stability in the case of no input, some of them considered the case of periodic input [6]–[10].

In [9], the contraction mapping technique was used to prove the existence of a stable periodic solution. In particular, it was shown that if the circle criterion in [3] is met, an asymptotically periodic input produces an asymptotically periodic output with the same period. On the other hand, the special case when the input signal is sinusoidal was considered by Miller and Michel [10]. They used the describing function method to predict the existence of a periodic solution and further showed that if the nonlinear feedback system satisfies some computable conditions, then it actually has a periodic solution which is nearly equal to the predicted one.

On the other hand, in many communication systems, such as modulators, mixers, and frequency converters, the frequency components of the input signal are generally incommensurate, and hence the differential equations describing the dynamic behavior of the communication system to the input signal are not periodic but almost periodic [14], [15]. Even in other practical systems such as parametric amplifiers, the corresponding differential equations can be almost periodic because of time-varying components. In this context, some authors recently have paid attention to the almost-periodic properties of the nonlinear feedback systems of the forms in Figs. 1 and 2. In fact, they are almost periodic when the input r and/or the nonlinearity η is almost periodic. Furthermore, even in the simple case when both the input r and the nonlinearity η are periodic but their periods are not commensurate, the nonlinear feedback systems in Figs. 1 and 2 are not periodic but almost periodic.

Recently, Sandberg [12] characterized the output responses of the nonlinear systems with asymptotically almost periodic input, whose input–output maps have approximately finite memory. In [13], he also showed that the nonlinear feedback system of the form in Fig. 1 with η independent of t has approximately finite memory if the circle criterion in [3] is met. Hence, it is the direct consequence of his results in [12] and [13] that if the circle criterion in [3] is satisfied, the output of the nonlinear feedback system in Fig. 1 with η independent of t to an asymptotically almost periodic input is also asymptotically almost periodic. The analyses in [12] and [13] were performed by an operator-theoretic approach.

On the other hand, there have been two different approaches to the feedback system analysis. One is the operator-theoretic approach based on the input–output maps of systems, and the other is the state-space approach based on the differential equations of systems. Although the two approaches are closely related, they are in general not equivalent and an analysis by both approaches will frequently yield more insight than can be gained from only one of the two approaches [11].

In this context, we take a state-space approach to the analysis of the nonlinear feedback systems in Figs. 1 and 2. Thereby, it is shown that Sandberg's result about the nonlinear feedback system in Fig. 1 with η independent of t can also be obtained by utilizing the well-known Meyer–Kalman–Yacubovich lemma [18]. It is further shown that Sandberg's result can be extended to the more general case of time-varying nonlinearities easily by using the state-space approach. Specifically speaking, it is shown that if r is almost periodic, η is almost periodic in t uniformly for u (y), and the circle criterion in [3] is satisfied, then the nonlinear feedback system in Fig. 1 (Fig. 2) has a unique almost periodic solution which is uniformly asymptotically stable in the large. It is also shown that the spectra of the almost periodic solution can be characterized by those of the input and the time-varying nonlinearity.

II. Main Results

The differential equation for the nonlinear feedback system of the form in Fig. 1 can be written as

$$\begin{aligned} \dot{x} &= Ax + b\eta(t,\, r-y) \\ y &= cx \end{aligned} \tag{1}$$

where $x \in R^n$, $y \in R$, $b \in R^n$, and $c^T \in R^n$. Here, the function η $[0, \infty) \times R \to R$ represents a memoryless nonlinearity and is said to *belong to the sector* $[\alpha, \beta]$ if

$$[\eta(t,\, u) - \alpha u][\eta(t,\, u) - \beta u] \le 0, \qquad \forall\, u \in R,\ \forall\, t \in [0, \infty).$$

On the other hand, η is said to *belong to the incremental sector* $[\alpha, \beta]_i$ if: 1) $\eta(t, x(t))$ is measurable when $x(t)$ is measurable; 2) $\eta(t, 0) = 0$, $\forall\, t \in [0, \infty)$; and 3)

$$\alpha \le \frac{\eta(t,\, u_1) - \eta(t,\, u_2)}{u_1 - u_2} \le \beta$$

for all real $u_1 \ne u_2$ and for all $t \in [0, \infty)$. A simple observation shows that if η belongs to the incremental sector $[\alpha, \beta]_i$, then it does to the sector $[\alpha, \beta]$.

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The authors are with the School of Electrical Engineering, Seoul National University, San 56-1, Shinrim-Dong, Kwanak-Ku, Seoul 151-742, Korea (e-mail: ijha@asri.snu.ac.kr).

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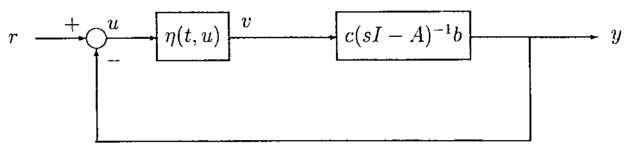

Fig. 1. Nonlinear feedback system I.

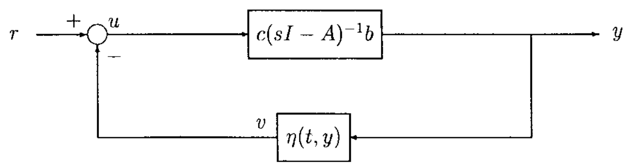

Fig. 2. Nonlinear feedback system II.

Suppose that the nonlinearity η in Fig. 1 belongs to the sector $[\alpha, \beta]$ and define the nonlinear function $\hat{\eta}$: $R \to R$ by

$$\hat{\eta}(t, u) \triangleq \eta(t, u) - \alpha u.$$

Then, $\hat{\eta}$ belongs to the sector $[0, \beta - \alpha]$ and the differential equation in (1) can be written as

$$\dot{x} = f(t, x, r) \tag{2}$$

where

$$f(t, x, r) \triangleq \hat{A}x + b\hat{\eta}(t, r - cx) + \alpha br$$
$$\hat{A} \triangleq A - \alpha bc.$$

When the input r of the differential equation in (2) is fixed, however, we simply write the differential equation in (2) as

$$\dot{x} = f_r(t, x) \triangleq f(t, x, r(t)) \tag{3}$$

for notational simplicity.

Now, we introduce some definitions of the boundedness and stability of solutions of differential equations, which can be found in [17], [18], and elsewhere. In what follows, the solution of the differential equation in (3) through a point $(t_0, x_0) \in R \times R^n$ is denoted by $x(t; t_0, x_0)$ and is assumed to be unique for each $(t_0, x_0) \in R \times R^n$. Let $\|x\|$ denote the norm of a vector $x \in R^n$. Then, the solution of the differential equation in (3) is said to be *ultimately bounded for bound* B, if for every solution $x(t; t_0, x_0)$ of the differential equation in (3), there exists a $\tau > 0$ such that $\|x(t; t_0, x_0)\| < B$ for all $t \geq t_0 + \tau$. In particular, when τ is independent of t_0, the solution of the differential equation in (3) is said to be *uniformly ultimately bounded*. As for the stability of solutions, the solution ϕ of the differential equation in (3) is said to be *uniformly stable*, if for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $t_0 \in [0, \infty)$ and $\|x_0 - \phi(t_0)\| < \delta(\epsilon)$, then $\|x(t; t_0, x_0) - \phi(t)\| < \epsilon$ for all $t \geq t_0$ and it is said to be *uniformly asymptotically stable in the large*, if: 1) it is uniformly stable and 2) for every $t_0 \in [0, \infty)$ and for every $x_0 \in R^n$, $\|x(t; t_0, x_0) - \phi(t)\| \to 0$ as $t \to \infty$.

Next, define a transfer function $\hat{g}(s)$ by

$$\hat{g}(s) \triangleq 1 + \kappa g(s)[1 + \alpha g(s)]^{-1} \tag{4}$$

where the constant κ and the transfer function g are defined by

$$\kappa \triangleq \beta - \alpha, \qquad g(s) \triangleq c(sI - A)^{-1}b$$

respectively. Then, $(\hat{A}, \kappa b, c, 1)$ is a realization of $\hat{g}(s)$ [not necessarily minimal since it is not assumed that (A, b) is controllable and (A, c) is observable]. Suppose that $\hat{g}(s)$ is strictly positive real. Then, the well-known Meyer–Kalman–Yacubovich lemma [18] implies that for any symmetric positive definite matrix $L \in R^{n \times n}$, there exist a symmetric positive matrix $P \in R^{n \times n}$, a vector $q^T \in R^n$, and $\gamma > 0$ such that

$$\begin{aligned} \hat{A}^T P + P\hat{A} &= -\gamma L - q^T q \\ b^T P &= \kappa c + \sqrt{2} q. \end{aligned} \tag{5}$$

Using the symmetric positive matrix P satisfying the above equations, we define two Lyapunov-like functions $V: R^n \to R$ and $W: R^n \times R^n \to R$ as follows:

$$V(x) \triangleq x^T P x, \qquad W(x, y) \triangleq V(x - y). \tag{6}$$

The following lemma shows that the circle criterion in [18] is a sufficient condition for the ultimate boundedness of the solution of the differential equation in (3) when the input of the nonlinear feedback system is bounded.

Lemma 1: Suppose that: 1) η belongs to the sector $[\alpha, \beta]$ and 2) r is bounded. Then, the solution of the differential equation in (3) is uniformly ultimately bounded if: 1) $\hat{A}$ is Hurwitz and 2) $\hat{g}(s)$ is strictly positive real.

Proof: In what follows, the minimum and maximum of the eigenvalues of a matrix $Q \in R^{n \times n}$ will be denoted by $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$, respectively. Take the time derivative of V in (6) along the solution of the differential equation in (3). Then

$$\dot{V} = x^T(\hat{A}^T P + P\hat{A})x + 2\alpha r b^T P x + 2\hat{\eta}(t, u) b^T P x$$

where $u \triangleq r - cx$. By (5), this implies that

$$\begin{aligned} \dot{V} =& -\gamma x^T L x - x^T q^T q x + 2\alpha r b^T P x + 2\kappa \hat{\eta}(t, u) y \\ &+ 2\sqrt{2}\hat{\eta}(t, u) q x \\ =& -\gamma x^T L x - \left[q x - \sqrt{2}\hat{\eta}(t, u)\right]^2 + 2\alpha r b^T P x \\ &+ 2\hat{\eta}(t, u)[\kappa y + \hat{\eta}(t, u)] \\ \leq& -\gamma \frac{\lambda_{\min}(L)}{\lambda_{\max}(P)} x^T P x + 2\alpha r_\infty \|b^T P\| \left[\frac{x^T P x}{\lambda_{\min}(P)}\right]^{1/2} \\ &+ 2\hat{\eta}(t, u)[\hat{\eta}(t, u) - \kappa u] + 2\kappa r \hat{\eta}(t, u) \\ \leq& -\hat{\gamma} V + 2\alpha r_\infty \|b^T P\| [\lambda_{\min}(P)]^{-(1/2)} V^{1/2} \\ &+ 2\kappa r \hat{\eta}(t, r - cx) \\ \leq& -\hat{\gamma} V + a_1 V^{1/2} + a_2 \end{aligned} \tag{7}$$

where

$$r_\infty \triangleq \sup_{t \geq 0} |r(t)|, \qquad \hat{\gamma} \triangleq \gamma \frac{\lambda_{\min}(L)}{\lambda_{\max}(P)}$$
$$\begin{aligned} a_1 \triangleq& \ 2\alpha r_\infty \|b^T P\| [\lambda_{\min}(P)]^{-(1/2)} \\ &+ 2\kappa^2 r_\infty \|c\| [\lambda_{\min}(P)]^{-(1/2)} \\ a_2 \triangleq& \ 2\kappa^2 r_\infty^2. \end{aligned}$$

The last two inequalities in (7) come from the fact that $\hat{\eta}$ belongs to the sector $[0, \kappa]$. Then, it is not difficult to show through simple arguments based on the above inequality that the solution of the differential equation in (3) is uniformly ultimately bounded. □

Before stating our main result, we need some other definitions, which can be found in [16] and [17]. A function g: $R \times R^m \to R^n$ is said to be *almost periodic in* t *uniformly for* x if for any $\epsilon > 0$ and for any compact set K in R^m, there exists a real number $l(\epsilon, K)$ such that every interval of length $l(\epsilon, K)$ in R contains at least one number τ satisfying

$$\|g(t + \tau, x) - g(t, x)\| \leq \epsilon$$

for all $t \in R$ and for all $x \in K$. Let $AP(R \times R^m, R^n)$ denote the space of all such functions g. In particular, $g \in AP(R, R^n)$ is simply said to be *almost periodic*. Recall that a function $v\colon R \to R^n$ is said to be T*-periodic* when there exists the smallest number $T > 0$ such that $v(t+T) = v(t)$, $\forall t \in R$. Then, it is obvious that all continuous periodic functions are almost periodic. For any $g \in AP(R \times R^m, R^n)$, let Λ_g be the set of real numbers ω such that

$$a_g(\omega, x) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_0^T g(t, x)e^{-j\omega t}\,dt$$

is not identically zero for $x \in R^n$. Then, the set Λ_g is countable [17]. The *module* $\overline{\Lambda}_g$ of g denotes the set defined as follows:

$$\left\{\omega\colon \omega = \sum_{j=1}^m k_j\omega_j;\ \omega_j \in \Lambda_g,\ k_j \geq 0,\ \text{and } m > 0 \text{ integers}\right\}.$$

Suppose that $r \in AP(R, R)$ and $\eta \in AP(R \times R, R)$. Define a function $\zeta\colon R \times R \to R$ by

$$\zeta(t, u) \triangleq \eta(t, r(t) - u). \tag{8}$$

Then

$$\zeta \in AP(R \times R, R). \tag{9}$$

The proof of this fact is similar to that given in [17, Th. 2.7] and is omitted. By (9) along with the definition of almost periodic functions, it follows that f_r in (3) is almost periodic in t uniformly for x.

Now, we are ready to state our main result.

Theorem 1: Suppose that: 1) η belongs to the incremental sector $[\alpha, \beta]_i$; 2) $\hat{A}$ is Hurwitz; and 3) $\hat{g}(s)$ is strictly positive real. If $r \in AP(R, R)$ and $\eta \in AP(R \times R, R)$, then the differential equation in (3) has an almost periodic solution ϕ which is uniformly asymptotically stable in the large. Furthermore

$$\overline{\Lambda}_\phi \subset \left\{\omega_1 + \omega_2\colon \omega_1 \in \overline{\Lambda}_r,\ \omega_2 \in \overline{\Lambda}_\eta\right\}.$$

Proof: A simple observation shows that r is bounded (see [16, p. 316]). From Lemma 1 it follows that the solution of the differential equation in (3) is uniformly ultimately bounded. In what follows, we simply denote the solutions of the differential equation in (3) with the initial conditions $x(t_0) = x_0$ and $x(t_0) = z_0$, respectively, by x and z. Then, the time derivative of the Lyapunov function W in (6) along the solutions x and z is given by

$$\begin{aligned}\dot{W}(x, z) &= (\dot{x} - \dot{z})^T P(x - z) + (x - z)^T P(\dot{x} - \dot{z})\\ &= (x - z)^T(\hat{A}^T P + P\hat{A})(x - z)\\ &\quad + 2[\hat{\eta}(t, u) - \hat{\eta}(t, v)]b^T P(x - z)\end{aligned} \tag{10}$$

where $u \triangleq r - cx$ and $v \triangleq r - cz$. On the other hand, it follows from (5) and the sector condition that

$$\begin{aligned}&2[\hat{\eta}(t, u) - \hat{\eta}(t, v)]b^T P[x - z]\\ &= 2[\hat{\eta}(t, u) - \hat{\eta}(t, v)]\left(\kappa c + \sqrt{2}q\right)(x - z)\\ &= -2\kappa(u - v)[\hat{\eta}(t, u) - \hat{\eta}(t, v)]\\ &\quad + 2\sqrt{2}[\hat{\eta}(t, u) - \hat{\eta}(t, v)]q(x - z)\\ &\leq -2[\hat{\eta}(t, u) - \hat{\eta}(t, v)]^2\\ &\quad + 2\sqrt{2}[\hat{\eta}(t, u) - \hat{\eta}(t, v)]q(x - z).\end{aligned} \tag{11}$$

Using (5), (10), and (11), we can then derive the following inequality:

$$\begin{aligned}\dot{W}(x, z) &\leq (x - z)^T(\hat{A}^T P + P\hat{A})(x - z)\\ &\quad + 2\sqrt{2}[\hat{\eta}(t, u) - \hat{\eta}(t, v)]q(x - z)\\ &\quad - 2[\hat{\eta}(t, u) - \hat{\eta}(t, v)]^2\\ &= -\gamma(x - z)^T L(x - z)\\ &\quad - \left[q(x - z) - \sqrt{2}(\hat{\eta}(t, u) - \hat{\eta}(t, v))\right]^2\\ &\leq -\hat{\gamma}W(x, z).\end{aligned} \tag{12}$$

So far, we have shown that all the hypotheses of [17, Th. 19.2] are satisfied. Hence, the differential equation in (3) has an almost periodic solution ϕ which is uniformly asymptotically stable in the large and $\overline{\Lambda}_\phi \subset \overline{\Lambda}_{f_r}$.

First, we show that the module of f_r can be characterized by the modules of the input r and the nonlinearity η. Note that $\overline{\Lambda}_\phi \subset \overline{\Lambda}_{f_r}$ implies $\overline{\Lambda}_\phi \subset \overline{\Lambda}_\zeta$ since $f_r(t, x) = Ax + b\zeta(t, cx)$. Therefore, it suffices to show that

$$\Lambda_\zeta \subset \Omega \triangleq \left\{\omega_1 + \omega_2\colon \omega_1 \in \overline{\Lambda}_r,\ \omega_2 \in \Lambda_\eta\right\}. \tag{13}$$

As shown later, this follows from the fact that for any $u \in R$ and for any $\epsilon > 0$, there exists an almost periodic function $h\colon R \to R$ such that

$$\sup_{t\in R}|\zeta(t, u) - h(t)| \leq \epsilon, \qquad \Lambda_h \subset \Omega. \tag{14}$$

To show the above fact, fix $u \in R$ and let K be any compact set in R such that $\{r(t) - u\colon t \in R\} \subset K$. Then, we can find an integer M, continuous functions $a_k\colon K \to R$, and constants ω_k, $k = 1, 2, \cdots, M$ such that

$$\begin{aligned}&\sup_{(t, v)\in R\times K}\left|\eta(t, v) - \sum_{k=1}^M a_k(v)e^{i\omega_k t}\right| \leq \frac{\epsilon}{2},\\ &\{\omega_k\colon k = 1, \cdots, M\} \subset \Lambda_\eta.\end{aligned} \tag{15}$$

The arguments used for the proofs of [19, Ch. 2, Ths. I and V] can be directly extended to the proof of this proposition [16]. Note also that for each $k = 1, \cdots, M$, the function $b_k\colon R \to R$ defined by $b_k(t) \triangleq a_k(r(t) - u)$ is almost periodic [19, p. 6] and $\overline{\Lambda}_{b_k} \subset \overline{\Lambda}_r$ [17, Th. 2.8]. This implies that there exist an integer N and constants b_{kl}, ω'_{kl}, $l = 1, \cdots, N$, $k = 1, \cdots, M$ such that for each $k = 1, \cdots, M$

$$\begin{aligned}&\sup_{t\in R}\left|b_k(t) - \sum_{l=1}^N b_{kl}e^{i\omega'_{kl}t}\right| \leq \frac{\epsilon}{2M},\\ &\{\omega'_{kl}\colon l = 1, \cdots, N\} \subset \overline{\Lambda}_r.\end{aligned} \tag{16}$$

Let $h(t) \triangleq \sum_{k=1}^M \sum_{l=1}^N b_{kl}e^{i\omega'_{kl}t}e^{i\omega_k t}$. Then, (14) is the direct consequence of (15) and (16).

Now, we show (13) by contradiction using (14). Suppose that (13) is false. Then, there exist $\overline{\omega} \in \Lambda_\zeta$ and $\overline{u} \in R$ such that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \zeta(t, \overline{u})e^{-j\overline{\omega}t}\,dt \neq 0, \qquad \overline{\omega} \notin \Omega. \tag{17}$$

Let $\{\epsilon_k\}$ be any sequence of positive real numbers which tends to zero as $k \to \infty$. By (14), we can then find a sequence of almost periodic functions $\{h_k\}$ such that for each $k = 1, \cdots$

$$\sup_{t\in R}|\zeta(t, \overline{u}) - h_k(t)| \leq \epsilon_k, \qquad \Lambda_{h_k} \subset \Omega. \tag{18}$$

By (18), we have

$$\lim_{k\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T [\zeta(t, \overline{u}) - h_k(t)]e^{-j\overline{\omega}t}\,dt = 0. \tag{19}$$

Note also that (17) and (19) imply that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T h_k(t)e^{-j\overline{\omega}t}\,dt \neq 0$$

for sufficiently large k. This is, however, contradictory to the initial assumption in (17). Hence, (13) is true. □

According to Theorem 1, the module of the unique almost periodic solution ϕ is characterized by the modules of the input r and the nonlinearity η. Moreover, it is included in the module of the input r when η is independent of t. As an illustrative example, consider the case when the input r is T_1 periodic and the nonlinearity η is

T_2-periodic, where there exists no integer pair (m_1, m_2) such that $m_1T_1 = m_2T_2$. Then

$$\overline{\Lambda}_\zeta \subset \left\{\frac{2\pi k_1}{T_1} + \frac{2\pi k_2}{T_2}\colon k_1,\, k_2 \text{ integers}\right\}.$$

By Theorem 1, if all the hypotheses in Theorem 1 are satisfied, the steady-state response of the nonlinear feedback system in Fig. 1 has only the harmonic components of the frequencies, $2\pi k_1/T_1 + 2\pi k_2/T_2,\ k_1,\ k_2 = 0, \pm 1, \pm 2, \cdots$.

It should also be noted that if (A, b) is controllable and (A, c) is observable, the conditions of Theorem 1 can be graphically verified as in the case of absolute stability [18].

Furthermore, Theorem 1 can be specialized to the case when the nonlinear feedback system is T-periodic. Suppose that r is T-periodic and η is T-periodic in t. Then, it is clear that f_r in (3) is T-periodic in t. Further, suppose that the conditions of Lemma 1 are satisfied. Then, [17, Th. 15.8] along with Lemma 1 guarantees that the differential equation in (3) has a T-periodic solution. According to Theorem 1, when η belongs to the incremental sector $[\alpha, \beta]_i$, the periodic solution is uniformly asymptotically stable in the large.

A continuous function $v\colon [0, \infty) \rightarrow R^n$ is said to be *asymptotically almost periodic* if $v = v_1 + v_2$, where v_1 is the restriction to $[0, \infty)$ of an element of $AP(R, R^n)$ and v_2 is a continuous function defined on $[0, \infty)$ which tends to zero as $t \rightarrow \infty$. Then v_1 is unique. Next, we consider the dynamic behavior of the system in (2) in which the input r is asymptotically almost periodic. For this, we need the following lemma in addition to Lemma 1 and Theorem 1.

Lemma 2: Suppose that Assumptions 1)–3) in Theorem 1 are satisfied. Let $r_1\colon [0, \infty) \rightarrow R$ be a bounded function and $r_2\colon [0, \infty) \rightarrow R$ be a continuous function which tends to zero as $t \rightarrow \infty$. Then, the solution x of the differential equation in (3) corresponding to the input $r = r_1 + r_2$ has the following property for any $t_0 \in [0, \infty)$ and for any $x_0,\, z_0 \in R^n$:

$$\|x(t;\, t_0,\, x_0) - z(t;\, t_0,\, z_0)\| \rightarrow 0 \quad \text{as } t \rightarrow \infty \tag{20}$$

where z denotes the solution of the differential equation in (3) corresponding to the input $r = r_1$.

Proof: In what follows, $x(t;\, t_0,\, x_0)$ and $z(t;\, t_0,\, z_0)$ will be denoted, respectively, by x and z for notational simplicity. Through some arguments similar to those used for the derivation of the inequality in (12), it can be shown that the time derivative of W in (6) along the solutions x and z satisfies the following inequality:

$$\dot{W}(x(t),\, z(t)) \le -\hat{\gamma}W(x(t),\, z(t)) + h(t) \tag{21}$$

where

$$h(t) \triangleq \{2\alpha b^T P[x(t) - z(t)] + 2\kappa[\hat{\eta}(t,\, r(t) - cx(t)) - \hat{\eta}(t,\, r(t) - cz(t))]\}\delta(t).$$

As a result, the following inequality holds

$$W(x(t),\, z(t)) \le \exp[-\hat{\gamma}(t - t_0)]W(x(t_0),\, z(t_0)) + \int_{t_0}^{t} \exp[-\hat{\gamma}(t - \tau)]h(\tau)\, d\tau. \tag{22}$$

On the other hand, it holds that

$$\lim_{t\rightarrow\infty} h(t) = 0$$

since, by Lemma 1, x and z are bounded. Therefore

$$\lim_{t\rightarrow\infty} \int_{t_0}^{t} \exp[-\hat{\gamma}(t - \tau)]h(\tau)\, d\tau = 0. \tag{23}$$

From (22) and (23), we finally see that

$$\lim_{t\rightarrow\infty} W(x(t),\, z(t)) = 0$$

and hence that (20) holds. □

Now, the following result is an immediate consequence of Theorem 1 and Lemma 2.

Corollary 1: Suppose that the assumptions 1)–3) in Theorem 1 are satisfied. Suppose further that $\eta \in AP(R \times R,\, R)$. Let $r_1\colon [0, \infty) \rightarrow R$ be a function in $AP(R, R)$ and $r_2\colon [0, \infty) \rightarrow R$ be a continuous function which tends to zero as $t \rightarrow \infty$. Then, the solution x of the differential equation in (3) corresponding to the asymptotically almost periodic input $r = r_1 + r_2$ is asymptotically almost periodic such that for any $t_0 \in [0, \infty)$ and for any $x_0 \in R^n$,

$$\|x(t;\, t_0,\, x_0) - \phi(t)\| \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

where the function $\phi\colon R \rightarrow R^n$ denotes the unique almost periodic solution of the differential equation in (3) corresponding to the input $r = r_1$. □

Many practical nonlinear feedback systems can also take the form in Fig. 2. The differential equation for the nonlinear feedback system of the form in Fig. 2 can be written as

$$\begin{aligned}\dot{x} &= Ax + b[r - \eta(t,\, y)]\\ y &= cx\end{aligned} \tag{24}$$

which is a little different from the differential equation in (1) for the nonlinear feedback system in Fig. 1. However, all results obtained so far for the nonlinear feedback system in Fig. 1 can be easily extended to the nonlinear feedback system in Fig. 2.

III. Conclusion

In this paper, we have presented some existence and stability theorems for almost periodic solutions in a special class of nonlinear feedback systems with sector nonlinearities. In fact, many practical systems can be described as a feedback connection of a linear part and a nonlinear element, as shown in Figs. 1 and 2. On the other hand, various external signals which are applied to practical systems can be regarded as periodic and/or almost periodic inputs. Hence, our theoretical results are of practical use. Finally, it should be noted that the results presented in this paper can be easily extended to a general class of vector nonlinearities which are not necessarily of diagonal form.

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