MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

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METR 2011–42 December 2011

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Theoretical Analysis of Sinc-Nyström Methods for Volterra Integral Equations

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December 2011

Abstract

In this paper, three theoretical results are presented on Sinc-Nyström methods for Volterra integral equations of the first and second kind that have been proposed by Muhammad et al. On their methods, the following two points have been desired to be improved: 1) their methods include a tuning parameter hard to be found unless the solution is given, and 2) convergence has not been proved in a precise sense. In a mathematically rigorous manner, we present 1) an implementable way to estimate the tuning parameter, and 2) a rigorous proof of the convergence with its rate explicitly revealed. Furthermore, we show 3) the resulting system of the schemes are well-conditioned. Numerical examples which support the theoretical results are also given.

1 Introduction

We are mainly concerned with Volterra integral equations of the second kind of the form:

$$u(t) - \int_a^t k(t,s)u(s)\,\mathrm{d}s = g(t), \quad a \leq t \leq b, \tag{1.1}$$

where $g(t)$ and $k(t,s)$ are given functions, and $u(t)$ is the solution to be determined. The equations have been employed as mathematical models in many fields, such as renewal processes [3], semi-conductor devices [14], wave phenomena [9], and population biology [5]. Because of the great importance in application, there have been considerable number of researches on the theory and numerical methods for the equations (see, for example, Linz [13], Kythe–Puri [11], Brunner [6], and references therein). We also consider Volterra integral equations of the first kind:

$$\int_a^t k(t,s)u(s)\,\mathrm{d}s = g(t), \quad a \leq t \leq b, \quad g(a) = 0, \tag{1.2}$$

by rewriting it as the second kind equation (1.1). There have also been researches for the first kind [1, 12].

If we turn our attention to numerical methods, Sinc numerical methods, which have been extensively studied by Stenger [29,30], seem to have become successful numerical tools, especially in the area of differential equations: for example, initial value problems of ordinary differential equations [7, 29, 30], boundary value problems of second-order ordinary differential equations [4, 28],

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and boundary value problems of fourth-order ordinary differential equations [16,27]. The typical convergence rate of these schemes is exponential, $\mathrm{O}(\exp(-c_1\sqrt{N}))$, which is much higher than the polynomial rate such as $\mathrm{O}(N^{-c})$. Moreover, it turned out that the rate can be improved to $\mathrm{O}(\exp(-c_2 N/\log N))$ by replacing the variable transformation used in those methods: IVP [31], BVP of second-order [19], BVP of fourth-order [20,21] (these transformations will be described later). It is also notable that such a rate can be attained even if functions to be approximated have end-point singularities [15,29,30,32].

As a natural extension of the studies, researches based on the Sinc numerical methods have also started in the area of integral equations. Numerical indefinite integration has been proposed by some authors [8,10,29] independently, and improved versions (by replacing the variable transformation) have been proposed [17,33]. By using the numerical indefinite integration, Muhammad et al. [18] have developed numerical schemes for Volterra integral equations of the second kind (1.1) and the first kind (1.2), where Nystöm's method is employed to discretize the equations (such a procedure is called a "Sinc-Nyström method"). They confirmed by numerical experiments that their schemes enjoy exponential convergence, $\mathrm{O}(\exp(-c_1\sqrt{N}))$ or $\mathrm{O}(\exp(-c_2 N/\log N))$, depending on the employed variable transformations. They have also given an error analysis (only for the latter convergence rate case) that their numerical solution u_N satisfies the following estimate (see also Theorem 3.1):

$$\sup_{t\in(a,\,b)}|u(t)-u_N(t)| \leq \left(C\sqrt{N}\|A_N^{-1}\|_2 + C'\right)\frac{\log N}{N}\exp(-c_2 N/\log N), \tag{1.3}$$

where A_N denotes the coefficient matrix of the resulting linear equations. Existence of A_N^{-1} and stability of its norm as N increases are suggested by their numerical experiments, but no rigorous proof has been given.

One of the purposes of this study is to fill this gap by giving an error analysis in the following form:

$$\max_{t\in[a,\,b]}|u(t)-u_N(t)| \leq C\frac{\log N}{N}\exp(-c_2 N/\log N), \tag{1.4}$$

which does not include any unestimated term and thus rigorously proves exponential convergence of the solution u_N. The key here is a technique developed in the analysis for Fredholm integral equations by the present authors [24]. We also proves the existence of A_N^{-1}, and boundedness of both $\|A_N\|_\infty$ and $\|A_N^{-1}\|_\infty$, which means the resulting system is well-conditioned.

Another purpose of this study, which should be more important in practice, is to introduce a way to estimate a tuning parameter 'd', which is required to set a mesh size h (see (2.15) and (2.17)). The choice is quite important since it substantially affects the convergence profile of the schemes (see numerical examples in Section 4.2). However, the optimal value of this parameter is in principle determined from the *unknown* true solution u, and so far there seems no practical way to find the optimal value. As a remedy, we show theoretically that the parameter can be estimated by investigating known functions k and g, instead of the solution u. Similar results have already been given for Fredholm integral equations [23,24,30].

This paper is organized as follows. Section 2 is a preliminary section that states numerical indefinite integration formulas by means of the Sinc approximation, and their convergence theorems. In Section 3, we review the schemes and error analysis by Muhammad et al. [18], and summarize new theoretical analyses in this paper (main results). The proofs will be given in Sections 5–7. Numerical experiments will be presented in Section 4, where the focus is mainly on the tuning parameter d. Finally in Section 8 we conclude this paper.

2 Preliminary: Sinc indefinite integration

2.1 Sinc indefinite integration on the real axis

Let us first consider the approximation on the entire real axis. A function approximation formula expressed as

$$F(\sigma) \approx \sum_{j=-N}^{N} F(jh)S(j,h)(\sigma), \quad \sigma \in \mathbb{R}, \tag{2.1}$$

is called the "Sinc approximation," where the basis function $S(j,h)$ is defined by

$$S(j,h)(x) = \frac{\sin[\pi(x/h - j)]}{\pi(x/h - j)}. \tag{2.2}$$

The mesh size h should be appropriately selected depending on N and smoothness of the function F (described in the subsequent convergence theorems). Using the Sinc approximation, Haber [8] has derived a numerical indefinite integration formula as follows:

$$\int_{-\infty}^{x} F(\sigma)\,\mathrm{d}\sigma \approx \sum_{j=-N}^{N} F(jh)\int_{-\infty}^{x} S(j,h)(\sigma)\,\mathrm{d}\sigma = \sum_{j=-N}^{N} F(jh)J(j,h)(x), \quad x \in \mathbb{R}, \tag{2.3}$$

where the basis function $J(j,h)$ is defined by

$$J(j,h)(x) = h\left\{\frac{1}{2} + \frac{1}{\pi}\operatorname{Si}[\pi(x/h - j)]\right\}. \tag{2.4}$$

Here, $\operatorname{Si}(x)$ is the so-called "sine integral" function, whose routine is available in some numerical libraries (IMSL, NAG, GSL, and so on). We call the formula (2.3) the "Sinc indefinite integration."

2.2 SE-Sinc indefinite integration and DE-Sinc indefinite integration

In the case of the finite interval (recall the target equations (1.1) and (1.2)), variable transformation is often utilized. The standard one is the "Single-Exponential transformation" defined by

$$s = \psi^{\mathrm{SE}}(\sigma) = \frac{b-a}{2}\tanh\left(\frac{\sigma}{2}\right) + \frac{b+a}{2}, \tag{2.5}$$

which maps $\sigma \in \mathbb{R}$ onto $s \in (a,\, b)$. We call it the "SE transformation" for short. The inverse function is

$$\sigma = \{\psi^{\mathrm{SE}}\}^{-1}(s) = \log\left(\frac{s-a}{b-s}\right). \tag{2.6}$$

Combining the SE transformation with the Sinc indefinite integration (2.3), Haber [8] has derived a indefinite integration formula on the finite interval as follows:

$$\begin{aligned}\int_a^t f(s)\,\mathrm{d}s &= \int_{-\infty}^{\{\psi^{\mathrm{SE}}\}^{-1}(t)} f(\psi^{\mathrm{SE}}(\sigma))\{\psi^{\mathrm{SE}}\}'(\sigma)\,\mathrm{d}\sigma \\ &\approx \sum_{j=-N}^{N} f(\psi^{\mathrm{SE}}(jh))\{\psi^{\mathrm{SE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t)), \quad t \in [a,\, b]. \end{aligned}\tag{2.7}$$

We call this approximation the "SE-Sinc indefinite integration."

For the purpose of improving the convergence rate, Muhammad–Mori [17] have proposed to replace the SE transformation with the Double-Exponential (DE) transformation:

$$s = \psi^{\mathrm{DE}}(\sigma) = \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh(\sigma)\right) + \frac{b+a}{2}, \tag{2.8}$$

whose inverse function is

$$\sigma = \{\psi^{\mathrm{DE}}\}^{-1}(s) = \log\left[\frac{1}{\pi}\log\left(\frac{s-a}{b-s}\right) + \sqrt{1 + \left\{\frac{1}{\pi}\log\left(\frac{s-a}{b-s}\right)\right\}^2}\,\right]. \tag{2.9}$$

The modified formula is expressed as

$$\int_a^t f(s)\,\mathrm{d}s \approx \sum_{j=-N}^{N} f(\psi^{\mathrm{DE}}(jh))\{\psi^{\mathrm{DE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(t)), \quad t \in [a,\, b]. \tag{2.10}$$

We call this approximation the "DE-Sinc indefinite integration."

2.3 Convergence theorems

In order to state the error analysis of the formulas above, let us introduce the following function spaces.

Definition 2.1. Let $\mathscr{D}$ be a bounded and simply-connected domain (or Riemann surface). Then $\mathbf{H}^\infty(\mathscr{D})$ denotes the family of functions f analytic on $\mathscr{D}$ such that the norm $\|f\|_{\mathbf{H}^\infty(\mathscr{D})}$ is finite, where

$$\|f\|_{\mathbf{H}^\infty(\mathscr{D})} = \sup_{z\in\mathscr{D}} |f(z)|. \tag{2.11}$$

Definition 2.2. Let α be a positive constant, and let $\mathscr{D}$ be a bounded and simply-connected domain (or Riemann surface) which satisfies $(a,\, b) \subset \mathscr{D}$. Then $\mathbf{L}_\alpha(\mathscr{D})$ denotes the family of functions $f \in \mathbf{H}^\infty(\mathscr{D})$ for which there exists a constant K such that for all z in $\mathscr{D}$

$$|f(z)| \leq K|Q(z)|^\alpha, \tag{2.12}$$

where the function Q is defined by $Q(z) = (z-a)(b-z)$.

In what follows, the domain $\mathscr{D}$ is supposed to be either

$$\psi^{\mathrm{SE}}(\mathscr{D}_d) = \{z = \psi^{\mathrm{SE}}(\zeta) : \zeta \in \mathscr{D}_d\} \quad \text{or} \quad \psi^{\mathrm{DE}}(\mathscr{D}_d) = \{z = \psi^{\mathrm{DE}}(\zeta) : \zeta \in \mathscr{D}_d\}, \tag{2.13}$$

which denotes the domain translated from the strip domain

$$\mathscr{D}_d = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < d\}, \tag{2.14}$$

for a positive constant d (see also Figures 1 and 2). Then convergence theorems of the SE/DE-Sinc indefinite integration can be stated as follows.

Theorem 2.3 (Okayama et al. [22, Theorem 2.7])**.** Let $(fQ) \in \mathbf{L}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$. Let N be a positive integer, and h be selected by the formula

$$h = \sqrt{\frac{\pi d}{\alpha N}}. \tag{2.15}$$

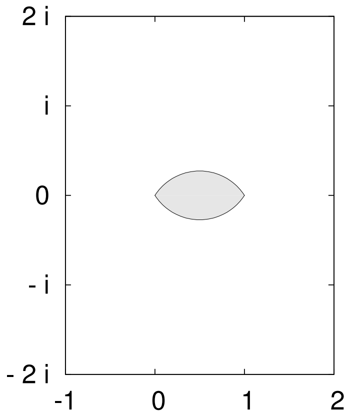

Figure 1. The domain $\psi^{\mathrm{SE}}(\mathscr{D}_1)$ where $(a, b) = (0, 1)$.

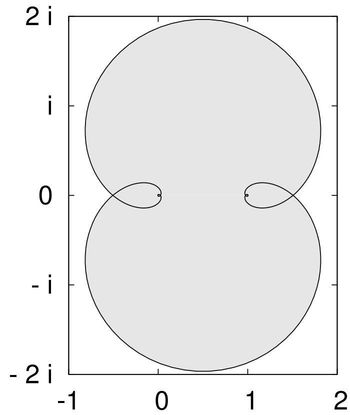

Figure 2. The domain $\psi^{\mathrm{DE}}(\mathscr{D}_1)$ where $(a, b) = (0, 1)$.

Then it holds that

$$\begin{aligned}&\max_{t\in[a,\,b]}\left|\int_a^t f(s)\,\mathrm{d}s - \sum_{j=-N}^{N} f(\psi^{\mathrm{SE}}(jh))\{\psi^{\mathrm{SE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t))\right|\\&\leq K(b-a)^{2\alpha-1}C_{\alpha,d}^{\mathrm{SE}}\,\mathrm{e}^{-\sqrt{\pi d\alpha N}},\end{aligned}\tag{2.16}$$

where K is the constant in the inequality (2.12), and $C_{\alpha,d}^{\mathrm{SE}}$ is a constant depending only on α and d.

Theorem 2.4 (Okayama et al. [22, Theorem 2.13])**.** Let $(fQ) \in \mathbf{L}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let N be a positive integer, and h be selected by the formula

$$h = \frac{\log(2dN/\alpha)}{N}.\tag{2.17}$$

Then it holds that

$$\begin{aligned}&\max_{t\in[a,\,b]}\left|\int_a^t f(s)\,\mathrm{d}s - \sum_{j=-N}^{N} f(\psi^{\mathrm{DE}}(jh))\{\psi^{\mathrm{DE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(t))\right|\\&\leq K(b-a)^{2\alpha-1}C_{\alpha,d}^{\mathrm{DE}}\frac{\log(2dN/\alpha)}{N}\exp\left[\frac{-\pi dN}{\log(2dN/\alpha)}\right],\end{aligned}\tag{2.18}$$

where K is the constant in the inequality (2.12), and $C_{\alpha,d}^{\mathrm{DE}}$ is a constant depending only on α and d.

Remark 2.5. On the SE-Sinc indefinite integration, Haber [8] has given an error estimate $\mathrm{O}(\sqrt{N}\,\mathrm{e}^{-\pi d\alpha N})$, but it is not sufficient for our purpose (see Remark 6.6). Similarly, the convergence rate of the DE-Sinc indefinite integration has been evaluated as $\mathrm{O}(\exp[-\pi dN/\log(2dN/\alpha)])$ in Muhammad–Mori [17], but the theorem above gives a stronger result.

Remark 2.6. In Theorems 2.3 and 2.4, the constants are stated more explicitly than usual. This is needed for the proofs of Lemmas 6.5 and 6.9. In view of (6.24) and (6.37), we notice that the function $F_i(t,z)Q(z)$ cannot be bounded by a constant, and should depend on h (and accordingly on N). We have to take care of the dependency on N, and the estimates in the above form are needed for the purpose.

3 Review of the existing results and the main results

First we review the schemes and theorems given by Muhammad et al. [18] in Sections 3.1 and 3.2. Then we summarize the main three results in Sections 3.3–3.5.

3.1 SE-Sinc-Nyström methods

Let us first consider Volterra integral equations of the second kind (1.1). Let $u \in \mathbf{H}^{\infty}(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ and $k(t,\cdot)Q(\cdot) \in \mathbf{L}_{\alpha}(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ for all $t \in [a,\, b]$. Then according to Theorem 2.3, the integral in the equation (1.1) can be approximated by the SE-Sinc indefinite integration as

$$\int_a^t k(t,s)u(s)\,\mathrm{d}s \approx \sum_{j=-N}^{N} k(t,\psi^{\mathrm{SE}}(jh))u(\psi^{\mathrm{SE}}(jh))\{\psi^{\mathrm{SE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t)), \tag{3.1}$$

where the mesh size h is selected by the formula (2.15). By this approximation we have a new equation:

$$u_N^{\mathrm{SE}}(t) = g(t) + \sum_{j=-N}^{N} k(t,\psi^{\mathrm{SE}}(jh))u_N^{\mathrm{SE}}(\psi^{\mathrm{SE}}(jh))\{\psi^{\mathrm{SE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t)). \tag{3.2}$$

In order to determine the approximated solution u_N^{SE}, we have to obtain the unknown coefficients

$$\boldsymbol{u}_m^{\mathrm{SE}} = [u_N^{\mathrm{SE}}(\psi^{\mathrm{SE}}(-Nh)),\, u_N^{\mathrm{SE}}(\psi^{\mathrm{SE}}(-(N-1)h)),\, \ldots,\, u_N^{\mathrm{SE}}(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}}, \tag{3.3}$$

where $m = 2N+1$. For this purpose, consider the discretization of the equation (3.2) at the SE-Sinc points:

$$t = \psi^{\mathrm{SE}}(ih), \quad i = -N,\, -N+1,\, \ldots,\, N. \tag{3.4}$$

Then the resulting linear system is written as

$$(I_m - V_m^{\mathrm{SE}})\boldsymbol{u}_m^{\mathrm{SE}} = \boldsymbol{g}_m^{\mathrm{SE}}, \tag{3.5}$$

where I_m denotes an $m \times m$ identity matrix and V_m^{SE} is an $m \times m$ matrix whose (i,j) element is defined by

$$(V_m^{\mathrm{SE}})_{ij} = k(\psi^{\mathrm{SE}}(ih),\psi^{\mathrm{SE}}(jh))\{\psi^{\mathrm{SE}}\}'(jh)J(j,h)(ih), \quad i,\, j = -N,\, -N+1,\, \ldots,\, N, \tag{3.6}$$

and $\boldsymbol{g}_m^{\mathrm{SE}} \in \mathbb{R}^m$ is defined by

$$\boldsymbol{g}_m^{\mathrm{SE}} = [g(\psi^{\mathrm{SE}}(-Nh)),\, g(\psi^{\mathrm{SE}}(-(N-1)h)),\, \ldots,\, g(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}}. \tag{3.7}$$

By solving the linear equation (3.5), the approximated solution u_N^{SE} is determined by (3.2).

Next let us consider Volterra integral equations of the first kind (1.2). If $k(t,t) \neq 0$ and both $k(\cdot,s)$ and g are differentiable, the equation can be reduced to the second kind equation:

$$u(t) - \int_a^t \tilde{k}(t,s)u(s)\,\mathrm{d}s = \tilde{g}(t), \tag{3.8}$$

where

$$\tilde{k}(t,s) = -\frac{1}{k(t,t)}\frac{\partial k(t,s)}{\partial t}, \qquad \tilde{g}(t) = \frac{g'(t)}{k(t,t)}. \tag{3.9}$$

In this case we can apply the same procedure as above. Let $u \in \mathbf{H}^{\infty}(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ and $\tilde{k}(t,\cdot)Q(\cdot) \in \mathbf{L}_{\alpha}(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ for all $t \in [a,\, b]$, and consider the approximated equation:

$$\tilde{u}_N^{\mathrm{SE}}(t) = \tilde{g}(t) + \sum_{j=-N}^{N} \tilde{k}(t,\psi^{\mathrm{SE}}(jh))\tilde{u}_N^{\mathrm{SE}}(\psi^{\mathrm{SE}}(jh))\{\psi^{\mathrm{SE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t)). \tag{3.10}$$

By discretizing this equation at the SE-Sinc points (3.4), we have a linear equation

$$(I_m - \tilde{V}_m^{\mathrm{SE}})\tilde{\boldsymbol{u}}_m^{\mathrm{SE}} = \tilde{\boldsymbol{g}}_m^{\mathrm{SE}}, \tag{3.11}$$

where $\tilde{\boldsymbol{u}}_m^{\mathrm{SE}}$, $\tilde{V}_m^{\mathrm{SE}}$, $\tilde{\boldsymbol{g}}_m^{\mathrm{SE}}$ are similarly defined to (3.3), (3.6), (3.7), respectively. By solving the linear equation (3.11), the approximated solution $\tilde{u}_N^{\mathrm{SE}}$ is determined by (3.10).

Although the above results have been mentioned in Muhammad et al. [18], the authors have mainly considered the DE-Sinc-Nyström methods described below, and no theoretical analysis has been given for the schemes above (the SE-Sinc-Nyström methods).

3.2 DE-Sinc-Nyström methods

Here we describe DE-Sinc-Nyström methods; this is done by replacing the SE transformation, which is used in the SE-Sinc-Nyström methods, with the DE transformation. Let us first consider the second kind equation (1.1). Let $u \in \mathbf{H}^{\infty}(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ and $k(t,\cdot)Q(\cdot) \in \mathbf{L}_{\alpha}(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for all $t \in [a,\, b]$. Then according to Theorem 2.4, the integral in the equation (1.1) can be approximated by the DE-Sinc indefinite integration as

$$\int_a^t k(t,s)u(s)\,\mathrm{d}s \approx \sum_{j=-N}^{N} k(t,\psi^{\mathrm{DE}}(jh))u(\psi^{\mathrm{DE}}(jh))\{\psi^{\mathrm{DE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(t)), \tag{3.12}$$

where the mesh size h is selected by the formula (2.17). By this approximation we have a new equation:

$$u_N^{\mathrm{DE}}(t) = g(t) + \sum_{j=-N}^{N} k(t,\psi^{\mathrm{DE}}(jh))u_N^{\mathrm{DE}}(\psi^{\mathrm{DE}}(jh))\{\psi^{\mathrm{DE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(t)). \tag{3.13}$$

In order to determine the approximated solution u_N^{DE}, we have to obtain unknown coefficients

$$\boldsymbol{u}_m^{\mathrm{DE}} = [u_N^{\mathrm{DE}}(\psi^{\mathrm{DE}}(-Nh)),\, u_N^{\mathrm{DE}}(\psi^{\mathrm{DE}}(-(N-1)h)),\, \ldots,\, u_N^{\mathrm{DE}}(\psi^{\mathrm{DE}}(Nh))]^{\mathrm{T}}, \tag{3.14}$$

where $m = 2N+1$. For this purpose, consider the discretization of the equation (3.13) at the DE-Sinc points:

$$t = \psi^{\mathrm{DE}}(ih), \quad i = -N,\, -N+1,\, \ldots,\, N. \tag{3.15}$$

Then the resulting linear system is written as

$$(I_m - V_m^{\mathrm{DE}})\boldsymbol{u}_m^{\mathrm{DE}} = \boldsymbol{g}_m^{\mathrm{DE}}, \tag{3.16}$$

where V_m^{DE} is an $m \times m$ matrix whose (i,j) element is defined by

$$(V_m^{\mathrm{DE}})_{ij} = k(\psi^{\mathrm{DE}}(ih),\psi^{\mathrm{DE}}(jh))\{\psi^{\mathrm{DE}}\}'(jh)J(j,h)(ih), \quad i,\, j = -N,\, -N+1,\, \ldots,\, N, \tag{3.17}$$

and $\boldsymbol{g}_m^{\mathrm{DE}} \in \mathbb{R}^m$ is defined by

$$\boldsymbol{g}_m^{\mathrm{DE}} = [g(\psi^{\mathrm{DE}}(-Nh)),\, g(\psi^{\mathrm{DE}}(-(N-1)h)),\, \ldots,\, g(\psi^{\mathrm{DE}}(Nh))]^{\mathrm{T}}. \tag{3.18}$$

By solving the linear equation (3.16), the approximated solution u_N^{DE} is determined by (3.13).

Next let us consider Volterra integral equations of the first kind (1.2) by reducing it to the second kind equation (3.8). Let $u \in \mathbf{H}^{\infty}(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ and $\tilde{k}(t,\cdot)Q(\cdot) \in \mathbf{L}_{\alpha}(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for all $t \in [a,\, b]$, and consider the approximated equation:

$$\tilde{u}_N^{\mathrm{DE}}(t) = \tilde{g}(t) + \sum_{j=-N}^{N} \tilde{k}(t,\psi^{\mathrm{DE}}(jh))\tilde{u}_N^{\mathrm{DE}}(\psi^{\mathrm{DE}}(jh))\{\psi^{\mathrm{DE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(t)). \tag{3.19}$$

By discretizing this equation at the DE-Sinc points (3.15), we have a linear equation

$$(I_m - \tilde{V}_m^{\mathrm{DE}})\tilde{\boldsymbol{u}}_m^{\mathrm{DE}} = \tilde{\boldsymbol{g}}_m^{\mathrm{DE}}, \tag{3.20}$$

where $\tilde{\boldsymbol{u}}_m^{\mathrm{DE}}$, $\tilde{V}_m^{\mathrm{DE}}$, $\tilde{\boldsymbol{g}}_m^{\mathrm{DE}}$ are defined similarly to (3.14), (3.17), (3.18), respectively. By solving the linear equation (3.20), the approximated solution $\tilde{u}_N^{\mathrm{DE}}$ is determined by (3.19).

On these schemes (DE-Sinc-Nyström methods), Muhammad et al. [18] have given the following error analyses.

Theorem 3.1 (Muhammad et al. [18, Theorem 3.2])**.** Let u be the solution of the equation (1.1). Let $u \in \mathbf{H}^\infty(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ and $k(t,\cdot)Q(\cdot) \in \mathbf{L}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for all $t \in [a,\, b]$. Then there exist constants C and C' independent of N such that

$$\sup_{a<t<b} |u(t) - u_N^{\mathrm{DE}}(t)| \leq \left(C\sqrt{N}\|(I_m - V_m^{\mathrm{DE}})^{-1}\|_2 + C'\right) \frac{\log N}{N} \exp\left[\frac{-\pi d N}{\log(2dN/\alpha)}\right]. \tag{3.21}$$

Theorem 3.2 (Muhammad et al. [18, Theorem 3.3])**.** Let u be the solution of the equation (1.2). Let $u \in \mathbf{H}^\infty(\psi^{\mathrm{DE}}(\mathscr{D}_d))$, let $k(t,t) \neq 0$, let $g(t)$ be differentiable, and let $\{\partial k(t,\cdot)/\partial t\}Q(\cdot) \in \mathbf{L}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ for all $t \in [a,\, b]$. Then there exist constants C and C' independent of N such that

$$\sup_{a<t<b} |u(t) - \tilde{u}_N^{\mathrm{DE}}(t)| \leq \left(C\sqrt{N}\|(I_m - \tilde{V}_m^{\mathrm{DE}})^{-1}\|_2 + C'\right) \frac{\log N}{N} \exp\left[\frac{-\pi d N}{\log(2dN/\alpha)}\right]. \tag{3.22}$$

3.3 Main result 1: Estimating the tuning parameter 'd'

In both the SE- and DE-Sinc-Nyström methods, the mesh size h is selected based on the parameters α and d (recall the formulas (2.15) and (2.17)). The former parameter α can be obtained by investigating the known function k (recall the assumption in Theorems 3.1 and 3.2). The latter parameter d is, however, not easy to know because it also depends on the *unknown* function u. Not only it is indispensable to launch the schemes, but it also substantially affects the performance (see Section 4.2). So far, however, no practical way to find the value has ever been known. In the previous study [18], the exact solution u seems to have been used to investigate d for numerical experiments, but in practical situations we cannot expect the solution u is known.

To remedy this issue, we present the following results. The proof will be given in Section 5.

Theorem 3.3. Let $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$ or $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_d)$. Let $g \in \mathbf{H}^\infty(\mathscr{D})$, let $k(z,\cdot)Q(\cdot) \in \mathbf{L}_\alpha(\mathscr{D})$, and let $k(\cdot,w)Q(w) \in \mathbf{H}^\infty(\mathscr{D})$ for all z, $w \in \mathscr{D}$. Then the equation (1.1) has a unique solution $u \in \mathbf{H}^\infty(\mathscr{D})$.

Theorem 3.4. Let $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$ or $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_d)$. Let $g' \in \mathbf{H}^\infty(\mathscr{D})$, let $\{\partial k(z,\cdot)/\partial z\}Q(\cdot) \in \mathbf{L}_\alpha(\mathscr{D})$, and let $\{\partial k(\cdot,w)/\partial z\}Q(w) \in \mathbf{H}^\infty(\mathscr{D})$ for all z, $w \in \mathscr{D}$. Furthermore let $1/k(z,z) \in \mathbf{H}^\infty(\mathscr{D})$. Then the equation (1.2) has a unique solution $u \in \mathbf{H}^\infty(\mathscr{D})$.

These theorems state that the parameter d of the *unknown* function u can be found from *known* functions k and g. Roughly speaking, by investigating the values of d of both k and g, we can choose the smaller one as d in the overall formula.

Remark 3.5. Strictly speaking, Theorems 3.3 and 3.4 give just a *safe* choice of d, and the optimal value might be larger than estimated. For example, let us consider

$$u(t) - \sqrt{1+(2t-1)^2}\int_0^t u(s)\,\mathrm{d}s = t^2 - \frac{t^3}{3}\sqrt{1+(2t-1)^2}, \quad 0 \leq t \leq 1. \tag{3.23}$$

The functions g and $k(\cdot,s)$ in this equation are not analytic at the points $z = (1 \pm \mathrm{i})/2$, and the parameter d can be taken to $d = \pi/2$ in the SE case at most, and to $d = \pi/6$ in the DE case. However the solution is $u(t) = t^2$, which is analytic on the whole complex plane. In this way, in several exceptional cases, the singularities of g and k might cancel, and the estimated value of d can be too moderate, i.e., be smaller than the optimal value.

3.4 Main result 2: Rigorous proof of the exponential convergence

In Theorems 3.1 and 3.2, there remained the N-dependent terms $\|(I_m - V_m^{\mathrm{DE}})^{-1}\|_2$ and $\|(I_m - \tilde{V}_m^{\mathrm{DE}})^{-1}\|_2$. In Muhammad et al. [18], it has been numerically confirmed that these terms remain low in all of their experiments, which suggests that their schemes converge exponentially. However, the boundedness of these terms has not been proved theoretically so far. Furthermore, in those theorems the existence of the inverse matrices $(I_m - V_m^{\mathrm{DE}})^{-1}$ and $(I_m - \tilde{V}_m^{\mathrm{DE}})^{-1}$, which is related to the feasibility of the schemes, is implicitly assumed and not proved.

In the present paper, we will rigorously prove the exponential convergence, providing the missing discussions described above. The proof will be given in Section 6.

Theorem 3.6 (SE, 2nd kind)**.** Let the assumptions in Theorem 3.3 be satisfied with $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$. Let $g \in C([a, b])$, and let $k(t, \cdot)Q(\cdot) \in C([a, b])$ and $k(\cdot, s)Q(s) \in C([a, b])$ for all $t, s \in [a, b]$. Then there exists a positive integer N_0 such that for all $N \geq N_0$ the coefficient matrix $(I_m - V_m^{\mathrm{SE}})$ is invertible. Furthermore, there exists a constant C for all $N \geq N_0$ such that

$$\max_{a \leq t \leq b} |u(t) - u_N^{\mathrm{SE}}(t)| \leq C\,\mathrm{e}^{-\sqrt{\pi d \alpha N}}. \tag{3.24}$$

Theorem 3.7 (DE, 2nd kind)**.** Let the assumptions in Theorem 3.3 be satisfied with $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_d)$. Let $g \in C([a, b])$, and let $k(t, \cdot)Q(\cdot) \in C([a, b])$ and $k(\cdot, s)Q(s) \in C([a, b])$ for all $t, s \in [a, b]$. Then there exists a positive integer N_0 such that for all $N \geq N_0$ the coefficient matrix $(I_m - V_m^{\mathrm{DE}})$ is invertible. Furthermore, there exists a constant C for all $N \geq N_0$ such that

$$\max_{a \leq t \leq b} |u(t) - u_N^{\mathrm{DE}}(t)| \leq C \frac{\log(2dN/\alpha)}{N} \exp\left[\frac{-\pi d N}{\log(2dN/\alpha)}\right]. \tag{3.25}$$

Remark 3.8. The assumption that g and kQ belong to $C([a, b])$ is just needed to guarantee the continuity at the endpoints $t = a, b$ since the domains $\psi^{\mathrm{SE}}(\mathscr{D}_d)$ and $\psi^{\mathrm{DE}}(\mathscr{D}_d)$ do not include the endpoints. The same remark applies to the following theorems.

Theorem 3.9 (SE, 1st kind)**.** Let the assumptions in Theorem 3.4 be satisfied with $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$. Let $g' \in C([a, b])$, let $\{\partial k(t, \cdot)/\partial t\}Q(\cdot) \in C([a, b])$, and let $\{\partial k(\cdot, s)/\partial t\}Q(s) \in C([a, b])$ for all $t, s \in [a, b]$. Furthermore let $1/k(t, t) \in C([a, b])$. Then there exists a positive integer N_0 such that for all $N \geq N_0$ the coefficient matrix $(I_m - \tilde{V}_m^{\mathrm{SE}})$ is invertible. Furthermore, there exists a constant C for all $N \geq N_0$ such that

$$\max_{a \leq t \leq b} |u(t) - \tilde{u}_N^{\mathrm{SE}}(t)| \leq C\,\mathrm{e}^{-\sqrt{\pi d \alpha N}}. \tag{3.26}$$

Theorem 3.10 (DE, 1st kind)**.** Let the assumptions in Theorem 3.4 be satisfied with $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_d)$. Let $g' \in C([a, b])$, let $\{\partial k(t, \cdot)/\partial t\}Q(\cdot) \in C([a, b])$, and let $\{\partial k(\cdot, s)/\partial t\}Q(s) \in C([a, b])$ for all $t, s \in [a, b]$. Furthermore let $1/k(t, t) \in C([a, b])$. Then there exists a positive integer N_0 such that for all $N \geq N_0$ the coefficient matrix $(I_m - \tilde{V}_m^{\mathrm{DE}})$ is invertible. Furthermore, there exists a constant C for all $N \geq N_0$ such that

$$\max_{a \leq t \leq b} |u(t) - \tilde{u}_N^{\mathrm{DE}}(t)| \leq C \frac{\log(2dN/\alpha)}{N} \exp\left[\frac{-\pi d N}{\log(2dN/\alpha)}\right]. \tag{3.27}$$

3.5 Main result 3: Analysis of conditions of the resulting systems

In Section 3.4, we have shown the invertibility of the coefficient matrices appearing in the schemes. In addition, we show stronger results: we will prove the boundedness of the condition numbers of the coefficient matrices as follows. The proof will be given in Section 7.

Theorem 3.11 (SE, 2nd kind)**.** Let the assumptions in Theorem 3.6 be satisfied with $\mathscr{D} = \psi^{\text{SE}}(\mathscr{D}_d)$. Then there exists a constant C_1 independent of N such that

$$\|(I_m - V_m^{\text{SE}})\|_\infty \leq C_1. \tag{3.28}$$

Furthermore there exists a constant C_2 independent of N such that for all $N \geq N_0$

$$\|(I_m - V_m^{\text{SE}})^{-1}\|_\infty \leq C_2, \tag{3.29}$$

where N_0 is the same integer as in Theorem 3.6.

Theorem 3.12 (DE, 2nd kind)**.** Let the assumptions in Theorem 3.7 be satisfied with $\mathscr{D} = \psi^{\text{DE}}(\mathscr{D}_d)$. Then there exists a constant C_1 independent of N such that

$$\|(I_m - V_m^{\text{DE}})\|_\infty \leq C_1. \tag{3.30}$$

Furthermore there exists a constant C_2 independent of N such that for all $N \geq N_0$

$$\|(I_m - V_m^{\text{DE}})^{-1}\|_\infty \leq C_2, \tag{3.31}$$

where N_0 is the same integer as in Theorem 3.7.

Theorem 3.13 (SE, 1st kind)**.** Let the assumptions in Theorem 3.9 be satisfied with $\mathscr{D} = \psi^{\text{SE}}(\mathscr{D}_d)$. Then there exists a constant C_1 independent of N such that

$$\|(I_m - \tilde{V}_m^{\text{SE}})\|_\infty \leq C_1. \tag{3.32}$$

Furthermore there exists a constant C_2 independent of N such that for all $N \geq N_0$

$$\|(I_m - \tilde{V}_m^{\text{SE}})^{-1}\|_\infty \leq C_2, \tag{3.33}$$

where N_0 is the same integer as in Theorem 3.9.

Theorem 3.14 (DE, 1st kind)**.** Let the assumptions in Theorem 3.10 be satisfied with $\mathscr{D} = \psi^{\text{DE}}(\mathscr{D}_d)$. Then there exists a constant C_1 independent of N such that

$$\|(I_m - \tilde{V}_m^{\text{DE}})\|_\infty \leq C_1. \tag{3.34}$$

Furthermore there exists a constant C_2 independent of N such that for all $N \geq N_0$

$$\|(I_m - \tilde{V}_m^{\text{DE}})^{-1}\|_\infty \leq C_2, \tag{3.35}$$

where N_0 is the same integer as in Theorem 3.10.

Remark 3.15. The matrix norm in the four theorems above is infinity-norm: $\|\cdot\|_\infty$, whereas 2-norm is used in Theorems 3.1 and 3.2. This is because of some technical reasons.

4 Numerical Examples

The main purpose of the numerical experiments in this section is to investigate how the performance of the Sinc-Nyström methods varies by the tuning parameter d (related to Section 3.3). Also the rate of convergence is also confirmed (related to Section 3.4). In addition we confirm that the resulting system is in fact well-conditioned by investigating the condition number (related to Section 3.5).

We used C++ with double precision arithmetic for implementation. GNU Scientific Library (GSL) was used for computing the sine integral function $\text{Si}(x)$, appearing in the basis function $J(j,h)$ defined by (2.4).

4.1 Estimating the tuning parameter 'd'

Throughout this section, we consider the following test equation:

$$u(t) = \frac{\sqrt{t}}{2}\left\{\frac{6}{1+3t^2} + \log(1+3t^2)\right\} - \int_0^t \sqrt{ts}\,u(s)\,\mathrm{d}s, \quad 0 \leq t \leq 1. \tag{4.1}$$

Let us first estimate the parameter d of the solution u by using Theorem 3.3, without using any information on the solution. Let $\pi_\epsilon = \pi - \epsilon$, where ϵ denotes an arbitrary small positive number. The functions g and k of the equation (4.1) satisfy the assumptions in Theorem 3.3 with $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_{2\pi_\epsilon/3})$ and $\alpha = 1$ in the SE case. In the DE case, let us set the following values:

$$p = \frac{\pi}{3\log 2}, \tag{4.2}$$

$$x = -\frac{\log 2}{\pi}\left[1 - \sqrt{\frac{1+5p^2+\sqrt{(1+5p^2)^2+(4p)^2}}{2}}\right], \tag{4.3}$$

$$y = \frac{2}{3}\left[1 - \sqrt{\frac{2}{1+5p^2+\sqrt{(1+5p^2)^2+(4p)^2}}}\right], \tag{4.4}$$

$$d_\epsilon = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right) - \epsilon. \tag{4.5}$$

Then the assumptions in Theorem 3.3 are fulfilled with $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_{d_\epsilon})$ and $\alpha = 1$. Hence, the parameter d can be estimated as $d = 2\pi_\epsilon/3 \simeq 2.09$ in the SE case, and $d = d_\epsilon \simeq 0.69$ in the DE case. The analytical solution of the equation (4.1) is $u(t) = 3\sqrt{t}/(1+3t^2)$, which actually belongs to $\mathbf{H}^\infty(\psi^{\mathrm{SE}}(\mathscr{D}_{2\pi_\epsilon/3}))$ and $\mathbf{H}^\infty(\psi^{\mathrm{DE}}(\mathscr{D}_{d_\epsilon}))$. In this way, the parameter d can be estimated by Theorem 3.3 without the solution.

4.2 How the performance in fact varies depending on the parameter

The parameter d estimated above is used for computing the mesh size h, defined by (2.15) or (2.17). We are here interested in how the performance of the schemes varies when d is selected *incorrectly*. To investigate it, we here define h as

$$h = \sqrt{\frac{2^r \pi_\epsilon \pi}{3N}} \tag{4.6}$$

in the SE case, and

$$h = \frac{\log(2^r d_\epsilon N)}{N} \tag{4.7}$$

in the DE case, and conduct numerical experiments with $r = -1$, 0, 1, 2, 3 ($r = 1$ is optimal in theory). The results are shown in Figures 3–6. We checked the error of the numerical solutions u_N^{SE} and u_N^{DE} on the equally-spaced 101 points over the interval [0, 1], and the maximum error among them are shown in those graphs. We observe that in the SE case (Figures 3 and 4), the performance gets remarkably worse as $|r-1|$ increases. The convergence rate in all cases looks consistently $\mathrm{O}(\exp(-c_1\sqrt{N}))$, although c_1 is different in each case. In the DE case (Figures 5 and 6), the convergence rate in all cases looks consistently $\mathrm{O}(\exp(-c_2 N/\log(N)))$, but the difference of the results is interesting. Although there is not so big difference like the SE case, the convergence rate looks improved as r decreases. Discussions about these observations are given from theoretical viewpoint in Section 4.4.

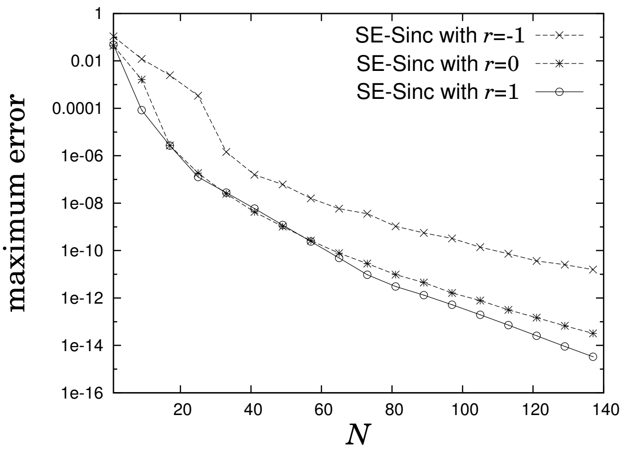

Figure 3. Performance of the SE-Sinc-Nyström method with h defined by (4.6), $r = -1$, 0, 1.

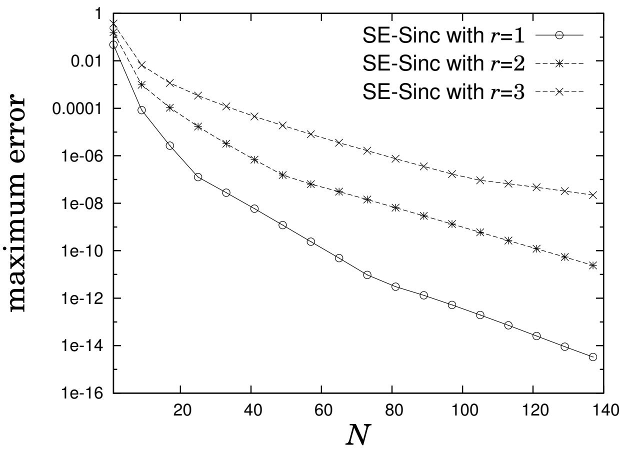

Figure 4. Performance of the SE-Sinc-Nyström method with h defined by (4.6), $r = 1$, 2, 3.

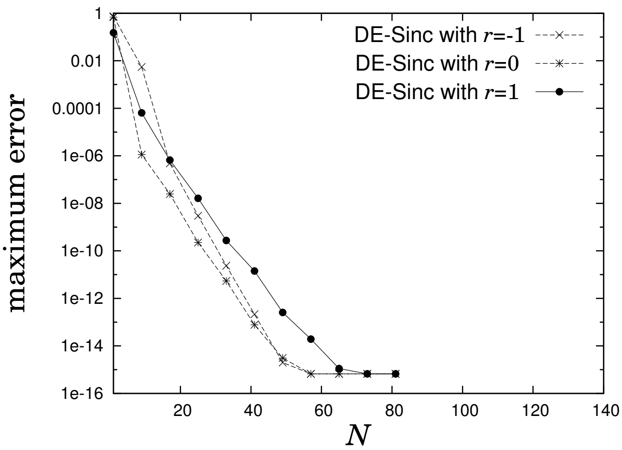

Figure 5. Performance of the DE-Sinc-Nyström method with h defined by (4.7), $r = -1$, 0, 1.

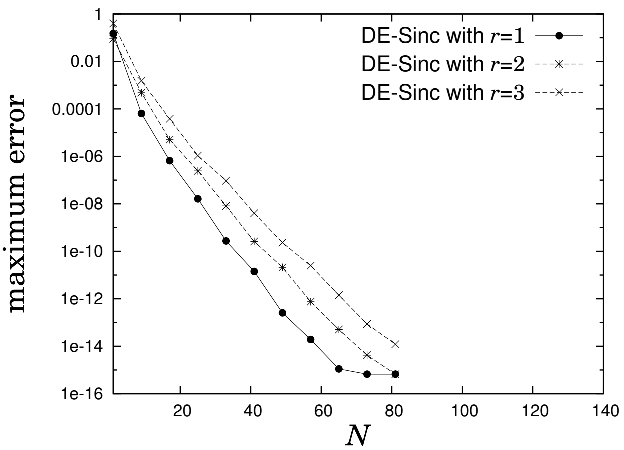

Figure 6. Performance of the DE-Sinc-Nyström method with h defined by (4.7), $r = 1,\ 2,\ 3$.

4.3 Checking the condition numbers of the coefficient matrices

We also investigate the condition numbers:

$$\|(I_m - V_m^{\mathrm{SE}})\|_\infty \|(I_m - V_m^{\mathrm{SE}})^{-1}\|_\infty \quad \text{and} \quad \|(I_m - V_m^{\mathrm{DE}})\|_\infty \|(I_m - V_m^{\mathrm{DE}})^{-1}\|_\infty, \tag{4.8}$$

in the case of $r = 1$. The results are shown in Figure 7, from which we can see the condition numbers remain quite low and bounded, at least in this range of N.

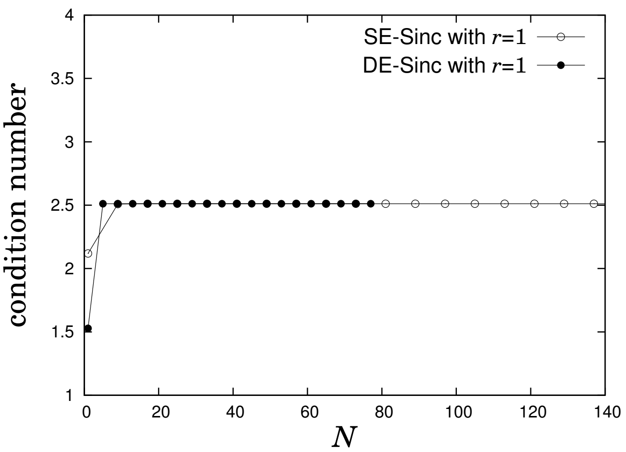

Figure 7. Condition number in the SE- and the DE-Sinc-Nyström method.

4.4 Discussions about the observed convergence rates

Let us roughly explain the convergence rates observed in Section 4.2 by analyzing the (SE/DE) Sinc indefinite integration, since it determines the performance of the Sinc-Nyström methods as shown in Section 6. The SE-Sinc indefinite integration has two main error terms: the "discretization error" and the "truncation error," which can be estimated as $\mathrm{O}(h\,\mathrm{e}^{-\pi d/h})$ and $\mathrm{O}(\mathrm{e}^{-\alpha N h})$ [29], respectively. Then we can estimate the whole error $E_{\text{SE-Sinc}}$ as

$$E_{\text{SE-Sinc}} \leq C_{\mathrm{D}} h\,\mathrm{e}^{-\pi d/h} + C_{\mathrm{T}}\,\mathrm{e}^{-\alpha N h}. \tag{4.9}$$

If we define the mesh size h as

$$h = \sqrt{\frac{c}{N}} \tag{4.10}$$

(recall (4.6), where $c = 2^{r-1}\pi d/\alpha$, and where $d = 2\pi_\epsilon/3$ and $\alpha = 1$), then we have

$$E_{\text{SE-Sinc}} \leq C_{\text{D}}\sqrt{\frac{c}{N}}\,\mathrm{e}^{-\pi d\sqrt{N/c}} + C_{\text{T}}\,\mathrm{e}^{-\alpha\sqrt{cN}}, \tag{4.11}$$

and we easily see that $c = \pi d/\alpha$ is optimal, giving the rate $\mathrm{O}(\mathrm{e}^{-\sqrt{\pi d\alpha N}})$. If c is bigger than it, then the first term limits the convergence rate to $\mathrm{O}(\mathrm{e}^{-\pi d\sqrt{N/c}}/\sqrt{N})$. Similarly, if c is smaller, then the rate becomes $\mathrm{O}(\mathrm{e}^{-\alpha\sqrt{cN}})$. Thus we can conclude that in all cases the convergence rate is actually consistently $\mathrm{O}(\exp(-c_1\sqrt{N}))$, as observed in the numerical experiments.

Next we consider the DE case. The DE-Sinc indefinite integration also has two main error terms stated above, but the truncation error is different from the SE case: $\mathrm{O}(\mathrm{e}^{-\frac{\pi}{2}\exp(Nh)})$ [22]. Therefore the whole error $E_{\text{DE-Sinc}}$ can be estimated as

$$E_{\text{DE-Sinc}} \leq C_{\text{D}} h\,\mathrm{e}^{-\pi d/h} + C_{\text{T}}\,\mathrm{e}^{-\frac{\pi}{2}\exp(Nh)}. \tag{4.12}$$

If we define the mesh size h as

$$h = \frac{\log(cN)}{N} \tag{4.13}$$

(recall (4.7), where $c = 2^r d/\alpha$, and where $d = d_\epsilon$ and $\alpha = 1$), then we have

$$\begin{aligned} E_{\text{DE-Sinc}} &\leq C_{\text{D}}\frac{\log(cN)}{N}\,\mathrm{e}^{-\pi dN/\log(cN)} + C_{\text{T}}\,\mathrm{e}^{-\frac{\pi}{2}cN} \\ &= \left\{C_{\text{D}} + C_{\text{T}}\frac{N}{\log(cN)}\,\mathrm{e}^{-\frac{\pi}{2}N(c-2d/\log(cN))}\right\}\frac{\log(cN)}{N}\,\mathrm{e}^{-\pi dN/\log(cN)}. \end{aligned} \tag{4.14}$$

Since the part in $\{\cdot\}$ can be bounded by a constant $\tilde{C}$ independent of N, the convergence rate is $\mathrm{O}(\log(cN)\,\mathrm{e}^{-\pi dN/\log(cN)}/N)$. From the estimate, we see that the rate can be improved by taking c as small as possible, which was roughly observed in the numerical experiments.

However, we should notice that if c is taken as too small, the part in $\{\cdot\}$ becomes large. Actually, in the small range of N in Figure 5, the error becomes larger as r decreases, which suggests the constant $\tilde{C}$ becomes larger. On the other hand, in Figure 6, the error in the case $r = 1$ (the standard case) is smaller than others in almost all the range of N. Considering the numerical experiments and the theoretical estimates above, we can say that the standard formula (2.17) ($r = 1$ in the experiments) is a well-balanced way to select h from both viewpoints: the convergence rate and the constant.

5 Proof of the main result 1

The idea here is to apply the standard contraction mapping theorem. Let $\mathscr{D} = \psi^{\text{SE}}(\mathscr{D}_d)$ or $\mathscr{D} = \psi^{\text{DE}}(\mathscr{D}_d)$, and let us introduce integral operators $\mathcal{V}$ and $\tilde{\mathcal{V}}$: $\mathbf{H}^\infty(\mathscr{D}) \to \mathbf{H}^\infty(\mathscr{D})$:

$$\mathcal{V}[f](t) = \int_a^t k(t,s)f(s)\,\mathrm{d}s, \qquad \tilde{\mathcal{V}}[f](t) = \int_a^t \tilde{k}(t,s)f(s)\,\mathrm{d}s, \tag{5.1}$$

where the kernels k and $\tilde{k}$ are assumed to satisfy the assumptions in Theorems 3.3 and 3.4, respectively (recall that $\tilde{k}$ is defined by (3.9)). These operators become contraction maps by multiplying repeatedly, as stated below.

Lemma 5.1. Let $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$ or $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_d)$. Let $f \in \mathbf{H}^\infty(\mathscr{D})$, let $k(z,\cdot)Q(\cdot) \in \mathbf{L}_\alpha(\mathscr{D})$, and let $k(\cdot,w)Q(w) \in \mathbf{H}^\infty(\mathscr{D})$ for all z, $w \in \mathscr{D}$. Then it holds for all positive integers n that

$$\|\mathcal{V}^n f\|_{\mathbf{H}^\infty(\mathscr{D})} \leq \frac{\{K(b-a)^{2\alpha-1}c_{\alpha,d}\}^n}{n!}\|f\|_{\mathbf{H}^\infty(\mathscr{D})}, \tag{5.2}$$

where K is a constant depending only on k, and $c_{\alpha,d}$ is a constant depending only on α and d.

Lemma 5.2. Let $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$ or $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_d)$. Let $f \in \mathbf{H}^\infty(\mathscr{D})$, let $\{\partial k(z,\cdot)/\partial z\}Q(\cdot) \in \mathbf{L}_\alpha(\mathscr{D})$, and let $\{\partial k(\cdot,w)/\partial z\}Q(w) \in \mathbf{H}^\infty(\mathscr{D})$ for all z, $w \in \mathscr{D}$. Furthermore let $1/k(z,z) \in \mathbf{H}^\infty(\mathscr{D})$. Then it holds for all positive integers n that

$$\|\tilde{\mathcal{V}}^n f\|_{\mathbf{H}^\infty(\mathscr{D})} \leq \frac{\{\tilde{K}(b-a)^{2\alpha-1}c_{\alpha,d}\}^n}{n!}\|f\|_{\mathbf{H}^\infty(\mathscr{D})}, \tag{5.3}$$

where $\tilde{K}$ is a constant depending only on k, and $c_{\alpha,d}$ is a constant depending only on α and d.

From these lemmas, the equation (1.1) and (1.2), which can be rewritten symbolically as $(\mathcal{I}-\mathcal{V})u = g$ and $(\mathcal{I}-\tilde{\mathcal{V}})u = \tilde{g}$, have a unique solution $u \in \mathbf{H}^\infty(\mathscr{D})$ by the contraction mapping theorem (note that $\mathbf{H}^\infty(\mathscr{D})$ is a Banach space [26, § 11.31]). Thus Theorems 3.3 and 3.4 follow.

In what follows we will prove Lemma 5.1, considering the SE and DE cases separately (proof of Lemma 5.2 goes in the same way, and we omit it).

The next lemma is needed for the SE case.

Lemma 5.3 (Okayama et al. [23, Lemma A.1])**.** Let d be a constant with $0 < d < \pi$ and let us define a function ψ_1 as

$$\psi_1(x) = \frac{1}{2}\tanh\left(\frac{x}{2}\right) + \frac{1}{2}. \tag{5.4}$$

Then there exists a constant c_1 depending only on d, such that for all $x \in \mathbb{R}$ and $y \in [-d,\, d]$

$$|Q(\psi^{\mathrm{SE}}(x + \mathrm{i}\, y))| \leq (b-a)^2 c_1 \psi_1'(x). \tag{5.5}$$

Since $0 \leq \psi_1(x) \leq 1$ holds, the following lemma is sufficient to establish Lemma 5.1.

Lemma 5.4. Let the assumptions in Lemma 5.1 be satisfied with $\mathscr{D} = \psi^{\mathrm{SE}}(\mathscr{D}_d)$. Then it holds for all positive integers n and $z \in \mathscr{D}$ that

$$|\mathcal{V}^n[f](z)| \leq \frac{\{K(b-a)^{2\alpha-1}c_1^\alpha \,\mathrm{B}(\psi_1(x),\alpha,\alpha)\}^n}{n!}\|f\|_{\mathbf{H}^\infty(\mathscr{D})}, \tag{5.6}$$

where $x = \mathrm{Re}[\{\psi^{\mathrm{SE}}\}^{-1}(z)]$, $\mathrm{B}(t,\alpha,\beta)$ is the incomplete beta function, K is the constant in the inequality (2.12) regarding $k(z,\cdot)Q(\cdot) \in \mathbf{L}_\alpha(\mathscr{D})$, and c_1 is a constant depending only on d.

Proof. Let $y = \mathrm{Im}[\{\psi^{\mathrm{SE}}\}^{-1}(z)]$ and accordingly $z = \psi^{\mathrm{SE}}(x + \mathrm{i}\, y)$. We show this lemma by induction. Consider the case $n = 1$ first. By the variable transformation $w = \psi^{\mathrm{SE}}(t + \mathrm{i}\, y)$, we have

$$\begin{aligned}
\mathcal{V}[f](z) &= \int_a^z k(z,w)f(w)\,\mathrm{d}w \\
&= \int_{-\infty}^x k(z,\psi^{\mathrm{SE}}(t+\mathrm{i}\,y))f(\psi^{\mathrm{SE}}(t+\mathrm{i}\,y))\{\psi^{\mathrm{SE}}\}'(t+\mathrm{i}\,y)\,\mathrm{d}t \\
&= \int_{-\infty}^x k(z,\psi^{\mathrm{SE}}(t+\mathrm{i}\,y))f(\psi^{\mathrm{SE}}(t+\mathrm{i}\,y))\frac{Q(\psi^{\mathrm{SE}}(t+\mathrm{i}\,y))}{b-a}\,\mathrm{d}t.
\end{aligned} \tag{5.7}$$

From $k(z,\cdot)Q(\cdot) \in \mathbf{L}_\alpha(\mathscr{D})$ and Lemma 5.3, it holds that

$$\begin{aligned}
|\mathcal{V}[f](z)| &\leq \frac{\|f\|_{\mathbf{H}^\infty(\mathscr{D})}K}{b-a}\int_{-\infty}^{x} |Q(\psi^{\mathrm{SE}}(t+\mathrm{i}\,y))|^\alpha \,\mathrm{d}t \\
&\leq \frac{\|f\|_{\mathbf{H}^\infty(\mathscr{D})}K}{b-a}\int_{-\infty}^{x} \{(b-a)^2 c_1 \psi_1'(t)\}^\alpha \,\mathrm{d}t \\
&= \|f\|_{\mathbf{H}^\infty(\mathscr{D})} K (b-a)^{2\alpha-1} c_1^\alpha \,\mathrm{B}(\psi_1(x),\alpha,\alpha). \qquad (5.8)
\end{aligned}$$

Hence the inequality (5.6) holds when $n=1$.

Next, assume the inequality (5.6) holds at n, and consider $n+1$. It holds that

$$\begin{aligned}
|\mathcal{V}[\mathcal{V}^n f](z)| &= \left|\int_{-\infty}^{x} k(z,\psi^{\mathrm{SE}}(t+\mathrm{i}\,y))\mathcal{V}^n[f](\psi^{\mathrm{SE}}(t+\mathrm{i}\,y))\frac{Q(\psi^{\mathrm{SE}}(t+\mathrm{i}\,y))}{b-a}\,\mathrm{d}t\right| \\
&\leq \int_{-\infty}^{x} \frac{K|Q(\psi^{\mathrm{SE}}(t+\mathrm{i}\,y))|^\alpha}{b-a}\frac{\{K(b-a)^{2\alpha-1}c_1^\alpha \,\mathrm{B}(\psi_1(t),\alpha,\alpha)\}^n}{n!}\|f\|_{\mathbf{H}^\infty(\mathscr{D})}\,\mathrm{d}t \\
&\leq \|f\|_{\mathbf{H}^\infty(\mathscr{D})}\{K(b-a)^{2\alpha-1}c_1^\alpha\}^{n+1}\int_{-\infty}^{x}\{\psi_1'(t)\}^\alpha \frac{\{\mathrm{B}(\psi_1(t),\alpha,\alpha)\}^n}{n!}\,\mathrm{d}t \\
&= \|f\|_{\mathbf{H}^\infty(\mathscr{D})}\{K(b-a)^{2\alpha-1}c_1^\alpha\}^{n+1}\frac{\{\mathrm{B}(\psi_1(x),\alpha,\alpha)\}^{n+1}}{(n+1)!}. \qquad (5.9)
\end{aligned}$$

This completes the proof. ■

Next let us consider the DE case. For this purpose the next lemma is needed.

Lemma 5.5 (Okayama et al. [23, Lemma A.4])**.** Let d be a constant with $0 < d < \pi/2$ and let us define a function ψ_2 as

$$\psi_2(x) = \frac{1}{2}\tanh\left(\frac{\pi\cos d}{2}\sinh(x)\right) + \frac{1}{2}. \qquad (5.10)$$

Then there exists a constant c_2 depending only on d, such that for all $x \in \mathbb{R}$ and $y \in [-d, d]$

$$|\pi\cosh(x+\mathrm{i}\,y)Q(\psi^{\mathrm{DE}}(x+\mathrm{i}\,y))| \leq (b-a)^2 c_2 \psi_2'(x). \qquad (5.11)$$

Since $0 \leq \psi_2(x) \leq 1$ holds, for Lemma 5.1, it is sufficient to prove the following lemma.

Lemma 5.6. Let the assumptions in Lemma 5.1 be satisfied with $\mathscr{D} = \psi^{\mathrm{DE}}(\mathscr{D}_d)$. Then it holds for all positive integers n and $z \in \mathscr{D}$ that

$$|\mathcal{V}^n[f](z)| \leq \frac{\{K(b-a)^{2\alpha-1}c_2' \,\mathrm{B}(\psi_2(x),\alpha,\alpha)\}^n}{n!}\|f\|_{\mathbf{H}^\infty(\mathscr{D})}, \qquad (5.12)$$

where $x = \mathrm{Re}[\{\psi^{\mathrm{DE}}\}^{-1}(z)]$, K is the constant in the inequality (2.12) regarding $k(z,\cdot)Q(\cdot) \in \mathbf{L}_\alpha(\mathscr{D})$, and $c_2' = (c_2^\alpha/\cos^{1-\alpha} d)$, where c_2 is a constant depending only on d.

Proof. Let $y = \mathrm{Im}[\{\psi^{\mathrm{DE}}\}^{-1}(z)]$ and accordingly $z = \psi^{\mathrm{DE}}(x+\mathrm{i}\,y)$. We show this lemma by induction. Consider the case $n=1$ first. By the variable transformation $w = \psi^{\mathrm{DE}}(t+\mathrm{i}\,y)$, we have

$$\begin{aligned}
\mathcal{V}[f](z) &= \int_a^z k(z,w)f(w)\,\mathrm{d}w \\
&= \int_{-\infty}^{x} k(z,\psi^{\mathrm{DE}}(t+\mathrm{i}\,y))f(\psi^{\mathrm{DE}}(t+\mathrm{i}\,y))\{\psi^{\mathrm{DE}}\}'(t+\mathrm{i}\,y)\,\mathrm{d}t \\
&= \int_{-\infty}^{x} k(z,\psi^{\mathrm{DE}}(t+\mathrm{i}\,y))f(\psi^{\mathrm{DE}}(t+\mathrm{i}\,y))\frac{\pi\cosh(t+\mathrm{i}\,y)Q(\psi^{\mathrm{DE}}(t+\mathrm{i}\,y))}{b-a}\,\mathrm{d}t. \qquad (5.13)
\end{aligned}$$

From $k(z,\cdot)Q(\cdot)\in\mathbf{L}_{\alpha}(\mathscr{D})$ and Lemma 5.5, it holds that

$$\begin{aligned}|\mathcal{V}[f](z)| &\leq \frac{\|f\|_{\mathbf{H}^{\infty}(\mathscr{D})}K}{b-a}\int_{-\infty}^{x}\pi|\cosh(t+\mathrm{i}\,y)|\cdot|Q(\psi^{\mathrm{DE}}(t+\mathrm{i}\,y))|^{\alpha}\,\mathrm{d}t\\ &\leq \frac{\|f\|_{\mathbf{H}^{\infty}(\mathscr{D})}K}{b-a}\int_{-\infty}^{x}\{\pi\cosh(t)\}^{1-\alpha}\{(b-a)^2c_2\psi_2'(t)\}^{\alpha}\,\mathrm{d}t\\ &= \|f\|_{\mathbf{H}^{\infty}(\mathscr{D})}K(b-a)^{2\alpha-1}\frac{c_2^{\alpha}}{\cos^{1-\alpha}d}\,\mathrm{B}(\psi_2(x),\alpha,\alpha). \end{aligned}\tag{5.14}$$

Hence the inequality (5.12) holds when $n=1$.

Next, assume the inequality (5.12) holds at n, and consider $n+1$. It holds that

$$\begin{aligned}&|\mathcal{V}[\mathcal{V}^n f](z)|\\ &=\left|\int_{-\infty}^{x}k(z,\psi^{\mathrm{DE}}(t+\mathrm{i}\,y))\mathcal{V}^n[f](\psi^{\mathrm{DE}}(t+\mathrm{i}\,y))\frac{\pi\cosh(t+\mathrm{i}\,y)Q(\psi^{\mathrm{DE}}(t+\mathrm{i}\,y))}{b-a}\,\mathrm{d}t\right|\\ &\leq\int_{-\infty}^{x}\frac{\pi|\cosh(t+\mathrm{i}\,y)|K|Q(\psi^{\mathrm{DE}}(t+\mathrm{i}\,y))|^{\alpha}}{b-a}\,\frac{\{K(b-a)^{2\alpha-1}c_2'\,\mathrm{B}(\psi_2(t),\alpha,\alpha)\}^n}{n!}\|f\|_{\mathbf{H}^{\infty}(\mathscr{D})}\,\mathrm{d}t\\ &\leq\|f\|_{\mathbf{H}^{\infty}(\mathscr{D})}\{K(b-a)^{2\alpha-1}c_2'\}^{n+1}\int_{-\infty}^{x}\{\pi\cosh(t)\cos d\}^{1-\alpha}\{\psi_2'(t)\}^{\alpha}\frac{\{\mathrm{B}(\psi_2(t),\alpha,\alpha)\}^n}{n!}\,\mathrm{d}t\\ &=\|f\|_{\mathbf{H}^{\infty}(\mathscr{D})}\{K(b-a)^{2\alpha-1}c_2'\}^{n+1}\frac{\{\mathrm{B}(\psi_2(x),\alpha,\alpha)\}^{n+1}}{(n+1)!}.\end{aligned}\tag{5.15}$$

This completes the proof. ■

6 Proof of the main result 2

In this section we prove Theorems 3.6 and 3.7 (Theorems 3.9 and 3.10 can be proved in the same way, and we omit it). Note that in this section $\|\cdot\|_{C([a,\,b])}$ is used as the norm, although $\|\cdot\|_{\mathbf{H}^{\infty}(\mathscr{D})}$ is used in the previous section. This is because we should evaluate the error not on the complex domain $\mathscr{D}$, but on the interval $[a,\,b]$. We write $\mathbf{C}=C([a,\,b])$ for short, and operators appearing in this section, such as $\mathcal{V}$, $\mathcal{V}_N^{\mathrm{SE}}$, $\mathcal{V}_N^{\mathrm{DE}}$, are assumed to map $\mathbf{C}$ onto $\mathbf{C}$.

6.1 Convergence analysis of the SE-Sinc-Nyström methods

First we prove the SE case, i.e., Theorem 3.6.

6.1.1 Sketch of the proof

Let us define an operator $\mathcal{V}_N^{\mathrm{SE}}$ as

$$\mathcal{V}_N^{\mathrm{SE}}[f](t)=\sum_{j=-N}^{N}k(t,\psi^{\mathrm{SE}}(jh))f(\psi^{\mathrm{SE}}(jh))\{\psi^{\mathrm{SE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t)),\tag{6.1}$$

which is the approximation of $\mathcal{V}f$ by the SE-Sinc indefinite integration. Then consider the following three equations:

$$(\mathcal{I}-\mathcal{V})u=g,\tag{6.2}$$

$$(\mathcal{I}-\mathcal{V}_N^{\mathrm{SE}})v=g,\tag{6.3}$$

$$(I_m-V_m^{\mathrm{SE}})\boldsymbol{c}_m=\boldsymbol{g}_m^{\mathrm{SE}}.\tag{6.4}$$

The first equation is nothing but (1.1), second is (3.2), and third is (3.5) itself. In order to prove the existence of $(I_m - V_m^{\mathrm{SE}})^{-1}$, which means feasibility of the scheme, we prove the following two claims:

1. the equation (6.4) is uniquely solvable if and only if the equation (6.3) is uniquely solvable (Lemma 6.1);
2. the equation (6.3) is uniquely solvable for all sufficiently large N (Lemma 6.7).

Combining Lemmas 6.1 and 6.7, we conclude Theorem 3.6.

6.1.2 Step 1: Equivalence of the solvability of (6.4) and (6.3)

Lemma 6.1. The following statements are equivalent:

(A) The equation (6.3) has a unique solution $v \in \mathbf{C}$.

(B) The equation (6.4) has a unique solution $\boldsymbol{c}_m \in \mathbb{R}^m$.

Proof. First we prove (A) $\Rightarrow$ (B). If we define $\boldsymbol{c}_m$ as $\boldsymbol{c}_m = [v(\psi^{\mathrm{SE}}(-Nh)), \ldots, v(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}}$, then clearly $\boldsymbol{c}_m$ is a solution of the equation (6.4). Next we prove the uniqueness. Let $\tilde{\boldsymbol{c}}_m = [\tilde{c}_{-N}, \ldots, \tilde{c}_N]^{\mathrm{T}}$ be another solution of the equation (6.4). Then let us define a function $\tilde{v} \in \mathbf{C}$ with the vector as

$$\tilde{v}(t) = g(t) + \sum_{j=-N}^{N} k(t, \psi^{\mathrm{SE}}(jh))\tilde{c}_j \{\psi^{\mathrm{SE}}\}'(jh) J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t)). \tag{6.5}$$

On the SE-Sinc points, $t = \psi^{\mathrm{SE}}(ih)$ $(i = -N, \ldots, N)$, it holds that

$$\tilde{v}(\psi^{\mathrm{SE}}(ih)) = g(\psi^{\mathrm{SE}}(ih)) + \sum_{j=-N}^{N} k(\psi^{\mathrm{SE}}(ih), \psi^{\mathrm{SE}}(jh))\tilde{c}_j \{\psi^{\mathrm{SE}}\}'(jh) J(j,h)(ih). \tag{6.6}$$

Furthermore, since $\tilde{\boldsymbol{c}}$ is a solution of the equation (6.4), it holds that

$$\tilde{c}_i = g(\psi^{\mathrm{SE}}(ih)) + \sum_{j=-N}^{N} k(\psi^{\mathrm{SE}}(ih), \psi^{\mathrm{SE}}(jh))\tilde{c}_j \{\psi^{\mathrm{SE}}\}'(jh) J(j,h)(ih). \tag{6.7}$$

Therefore $\tilde{v}(\psi^{\mathrm{SE}}(ih)) = \tilde{c}_i$, and from this the equation (6.6) can be rewritten as $(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})\tilde{v} = g$, which shows $\tilde{v}$ is a solution of the equation (6.3). By assumption (A), $v = \tilde{v}$ holds, and from which we have $\boldsymbol{c}_m = \tilde{\boldsymbol{c}}_m$. This shows the desired uniqueness.

Next we prove (B) $\Rightarrow$ (A). Let us define a function $v \in \mathbf{C}$ as

$$v(t) = g(t) + \sum_{j=-N}^{N} k(t, \psi^{\mathrm{SE}}(jh)) c_j \{\psi^{\mathrm{SE}}\}'(jh) J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t)). \tag{6.8}$$

Then by the same argument as above, $v(\psi^{\mathrm{SE}}(ih)) = c_i$ holds, from which we have $(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})v = g$. Therefore the equation (6.3) has a solution. Next we prove the uniqueness. Let $\tilde{v}$ be another solution of the equation (6.3). By using $\tilde{v}$, let us define $\tilde{\boldsymbol{c}}_m$ as $\tilde{\boldsymbol{c}}_m = [\tilde{v}(\psi^{\mathrm{SE}}(-Nh)), \ldots, \tilde{v}(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}}$. Clearly $\tilde{\boldsymbol{c}}_m$ is a solution of the equation (6.4), and by assumption (B), $\boldsymbol{c}_m = \tilde{\boldsymbol{c}}_m$ holds. Therefore $\tilde{v}(\psi^{\mathrm{SE}}(jh)) = c_j$ holds, and from this the equation $(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})\tilde{v} = g$ can be rewritten as

$$\tilde{v}(t) = g(t) + \sum_{j=-N}^{N} k(t, \psi^{\mathrm{SE}}(jh)) c_j \{\psi^{\mathrm{SE}}\}'(jh) J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t)). \tag{6.9}$$

Comparing the right hand side of (6.8) and (6.9), we conclude $v = \tilde{v}$, which shows the desired uniqueness. ■

6.1.3 Step 2: Solvability of the equation (6.3) for all sufficiently large N

For the analysis of Nyström methods, the next theorem is generally used.

Theorem 6.2 (Atkinson [2, Theorem 4.1.1])**.** Assume the following four conditions:

1. Operators $\mathcal{X}$ and $\mathcal{X}_n$ are bounded operators on $\mathbf{C}$ to $\mathbf{C}$.
2. The operator $(\mathcal{I} - \mathcal{X}) : \mathbf{C} \to \mathbf{C}$ has a bounded inverse $(\mathcal{I} - \mathcal{X})^{-1} : \mathbf{C} \to \mathbf{C}$.
3. The operator $\mathcal{X}_n$ is compact on $\mathbf{C}$.
4. The following inequality holds:

$$\|(\mathcal{X} - \mathcal{X}_n)\mathcal{X}_n\|_{\mathcal{L}(\mathbf{C},\mathbf{C})} < \frac{1}{\|(\mathcal{I} - \mathcal{X})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}}. \tag{6.10}$$

Then $(\mathcal{I} - \mathcal{X}_n)^{-1}$ exists as a bounded operator on $\mathbf{C}$ to $\mathbf{C}$, with

$$\|(\mathcal{I} - \mathcal{X}_n)^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})} \leq \frac{1 + \|(\mathcal{I} - \mathcal{X})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|\mathcal{X}_n\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}}{1 - \|(\mathcal{I} - \mathcal{X})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|(\mathcal{X} - \mathcal{X}_n)\mathcal{X}_n\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}}. \tag{6.11}$$

Furthermore, if $(\mathcal{I} - \mathcal{X})u = g$ and $(\mathcal{I} - \mathcal{X}_n)v = g$, then

$$\|u - v\|_{\mathbf{C}} \leq \|(\mathcal{I} - \mathcal{X}_n)^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|(\mathcal{X} - \mathcal{X}_n)u\|_{\mathbf{C}}. \tag{6.12}$$

In what follows we show the four conditions are fulfilled with $\mathcal{X} = \mathcal{V}$ and $\mathcal{X}_n = \mathcal{V}_N^{\text{SE}}$, under the assumptions in Theorem 3.6. The condition 1 clearly holds. The condition 2 is well-known (in fact we can prove it in the same way as in Section 5 by using the contraction mapping theorem). The condition 3 immediately follows from the Arzelá–Ascoli theorem. The main difficulty of this project lies in the condition 4. For this purpose we need a bound of the basis function $J(j,h)(x) = \int_{-\infty}^{x} S(j,h)(t)\,\mathrm{d}t$ as follows.

Lemma 6.3 (Stenger [29, Lemma 3.6.5])**.** For all $x \in \mathbb{R}$ it holds that

$$|J(j,h)(x)| \leq 1.1h. \tag{6.13}$$

This result can be extended to the complex plane as follows.

Lemma 6.4. For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$ it holds that

$$|J(j,h)(x + \mathrm{i}\,y)| \leq \frac{5h}{\pi} \cdot \frac{\sinh(\pi y/h)}{\pi y/h}. \tag{6.14}$$

Proof. We split the integral path as

$$J(j,h)(x + \mathrm{i}\,y) = \int_{-\infty}^{x+\mathrm{i}\,y} S(j,h)(\zeta)\,\mathrm{d}\zeta = \int_{-\infty}^{x} S(j,h)(\xi)\,\mathrm{d}\xi + \int_{0}^{y} S(j,h)(x + \mathrm{i}\,\eta)\,\mathrm{d}\eta, \tag{6.15}$$

and evaluate the two terms one by one. From Lemma 6.3, the first term can be bounded as

$$\left|\int_{-\infty}^{x} S(j,h)(\xi)\,\mathrm{d}\xi\right| \leq 1.1h < \frac{7h}{2\pi} \leq \frac{7h}{2\pi} \cdot \frac{\sinh(\pi y/h)}{\pi y/h}. \tag{6.16}$$

Next we evaluate the second term. Notice that the following inequality

$$|S(j,h)(x + \mathrm{i}\,\eta)| = \frac{h}{2\pi}\left|\int_{-\pi/h}^{\pi/h} \mathrm{e}^{\eta t + \mathrm{i}\,(jh - x)t}\,\mathrm{d}t\right| \leq \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \mathrm{e}^{\eta t}\,\mathrm{d}t = \frac{\sinh(\pi\eta/h)}{\pi\eta/h} \tag{6.17}$$

holds, and furthermore

$$\int_0^{|r|} \frac{\sinh t}{t}\,\mathrm{d}t \leq \frac{3}{2}\frac{\sinh r}{r} \tag{6.18}$$

holds for all $r \in \mathbb{R}$. Then we have

$$\begin{aligned}\left|\int_0^{y} S(j,h)(x+\mathrm{i}\,\eta)\,\mathrm{d}\eta\right| &\leq \int_0^{|y|} \frac{\sinh(\pi\eta/h)}{\pi\eta/h}\,\mathrm{d}\eta \\ &= \frac{h}{\pi}\int_0^{\pi|y|/h} \frac{\sinh t}{t}\,\mathrm{d}t \\ &\leq \frac{3h}{2\pi}\cdot\frac{\sinh(\pi y/h)}{\pi y/h}, \end{aligned} \tag{6.19}$$

which completes the proof. ■

By using this lemma we can prove the convergence of the term $\|(\mathcal{V}-\mathcal{V}_N^{\mathrm{SE}})\mathcal{V}_N^{\mathrm{SE}}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}$ as below.

Lemma 6.5. Let k satisfy the assumptions in Theorem 3.6. Then there exists a constant C independent of N such that

$$\|(\mathcal{V}-\mathcal{V}_N^{\mathrm{SE}})\mathcal{V}_N^{\mathrm{SE}}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})} \leq Ch, \tag{6.20}$$

where h is the mesh size defined by (2.15).

Proof. We show that there exists a constant C independent of N and f such that

$$\|(\mathcal{V}-\mathcal{V}_N^{\mathrm{SE}})\mathcal{V}_N^{\mathrm{SE}} f\|_{\mathbf{C}} \leq C\|f\|_{\mathbf{C}} h \tag{6.21}$$

holds for all $f \in \mathbf{C}$. Let us define a function $F_i(t,s)$ as

$$F_i(t,s) = k(t,s)k(s,\psi^{\mathrm{SE}}(ih))Q(\psi^{\mathrm{SE}}(ih))J(i,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t)). \tag{6.22}$$

Then $(\mathcal{V}-\mathcal{V}_N^{\mathrm{SE}})\mathcal{V}_N^{\mathrm{SE}} f$ can be rewritten as

$$\begin{aligned}&(\mathcal{V}-\mathcal{V}_N^{\mathrm{SE}})\mathcal{V}_N^{\mathrm{SE}}[f](t) \\ &= \sum_{i=-N}^{N} \frac{f(\psi^{\mathrm{SE}}(ih))}{b-a}\left\{\int_a^t F_i(t,s)\,\mathrm{d}s - \sum_{j=-N}^{N} F_i(t,\psi^{\mathrm{SE}}(jh))\{\psi^{\mathrm{SE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t))\right\}.\end{aligned} \tag{6.23}$$

First we evaluate the part in $\{\,\cdot\,\}$. From $k(t,\cdot)Q(\cdot) \in \mathbf{L}_\alpha(\psi^{\mathrm{SE}}(\mathscr{D}_d))$ and Lemma 6.4, it holds that

$$|F_i(t,z)Q(z)| \leq K|Q(z)|^\alpha KQ^\alpha(\psi^{\mathrm{SE}}(ih))\frac{5h}{\pi}\frac{\sinh(\pi d/h)}{\pi d/h} \tag{6.24}$$

for all integers i and $t \in [a,\,b]$ and $z \in \psi^{\mathrm{SE}}(\mathscr{D}_d)$. Therefore $F_i(t,\cdot)$ satisfies the assumptions of Theorem 2.3, from which we have

$$\begin{aligned}&\left|\int_a^t F_i(t,s)\,\mathrm{d}s - \sum_{j=-N}^{N} F_i(t,\psi^{\mathrm{SE}}(jh))\{\psi^{\mathrm{SE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t))\right| \\ &\leq \left\{K^2Q^\alpha(\psi^{\mathrm{SE}}(ih))\frac{5h}{\pi}\frac{\sinh(\pi d/h)}{\pi d/h}\right\}(b-a)^{2\alpha-1}C_{\alpha,d}^{\mathrm{SE}}\,\mathrm{e}^{-\sqrt{\pi d\alpha N}} \\ &= \frac{5K^2Q^\alpha(\psi^{\mathrm{SE}}(ih))(b-a)^{2\alpha-1}C_{\alpha,d}^{\mathrm{SE}}}{\pi^2 d}\cdot h^2\left[\sinh(\pi d/h)\,\mathrm{e}^{-\pi d/h}\right].\end{aligned} \tag{6.25}$$

The last equality holds from the formula (2.15). Furthermore, $|\sinh(\pi d/h)\,\mathrm{e}^{-\pi d/h}| \leq 1/2$ holds, which implies

$$
\begin{aligned}
|(\mathcal{V}-\mathcal{V}_N^{\mathrm{SE}})\mathcal{V}_N^{\mathrm{SE}}[f](t)| &\leq \sum_{i=-N}^{N} \frac{|f(\psi^{\mathrm{SE}}(ih))|}{b-a} \frac{5K^2 Q^{\alpha}(\psi^{\mathrm{SE}}(ih))(b-a)^{2\alpha-1} C_{\alpha,d}^{\mathrm{SE}}}{\pi^2 d} \cdot \frac{h^2}{2} \\
&\leq \|f\|_{\mathbf{C}} \frac{5K^2 (b-a)^{2\alpha-2} C_{\alpha,d}^{\mathrm{SE}}}{2\pi^2 d} h \left\{ h \sum_{i=-N}^{N} Q^{\alpha}(\psi^{\mathrm{SE}}(ih)) \right\}. \qquad (6.26)
\end{aligned}
$$

The term in $\{\,\cdot\,\}$ is uniformly bounded since it converges to $(b-a)^{2\alpha}\,\mathrm{B}(\alpha,\alpha)$. This shows the desired inequality (6.21). ■

Remark 6.6. In this proof we used Theorem 2.3, which says the convergence rate is $\mathrm{O}(\mathrm{e}^{-\sqrt{\pi d \alpha N}})$, to obtain the evaluation (6.25). However, as stated in Remark 2.5, if we employ the existing results $\mathrm{O}(\sqrt{N}\,\mathrm{e}^{-\sqrt{\pi d \alpha N}})$, we cannot prove the desired convergence, since $\|(\mathcal{V}-\mathcal{V}_N^{\mathrm{SE}})\mathcal{V}_N^{\mathrm{SE}}\| \sim \sqrt{N} h \sim$ const. > 0.

Thus all of the conditions 1–4 in Theorem 6.2 are fulfilled, and the next lemma follows.

Lemma 6.7. Suppose that the assumptions in Theorem 3.6 are fulfilled. Then there exists a positive integer N_0 such that for all $N \geq N_0$ the equation (6.3) has a unique solution $v \in \mathbf{C}$. Furthermore there exists a constant C independent of N such that for all $N \geq N_0$

$$
\|u - v\|_{\mathbf{C}} \leq C \|\mathcal{V}u - \mathcal{V}_N^{\mathrm{SE}} u\|_{\mathbf{C}}. \qquad (6.27)
$$

Proof. We have already confirmed the assumptions in Theorem 6.2, and what is left here is to show the boundedness of the term $\|(\mathcal{I}-\mathcal{V}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}$ (notice the inequality (6.12)). From the inequality (6.11), the claim follows if we prove that the term $\|\mathcal{V}_N^{\mathrm{SE}}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}$ is uniformly bounded. From Lemma 6.3 and $k(t,\cdot)Q(\cdot) \in \mathbf{L}_{\alpha}(\psi^{\mathrm{SE}}(\mathscr{D}_d))$, we have

$$
\begin{aligned}
|\mathcal{V}_N^{\mathrm{SE}}[f](t)| &= \left| \sum_{j=-N}^{N} k(t,\psi^{\mathrm{SE}}(jh)) f(\psi^{\mathrm{SE}}(jh)) \frac{Q(\psi^{\mathrm{SE}}(jh))}{b-a} J(j,h)(\{\psi^{\mathrm{SE}}\}^{-1}(t)) \right| \\
&\leq \frac{1.1 K \|f\|_{\mathbf{C}}}{b-a} \left\{ h \sum_{j=-N}^{N} Q^{\alpha}(\psi^{\mathrm{SE}}(jh)) \right\} \qquad (6.28)
\end{aligned}
$$

for all $f \in \mathbf{C}$. The term in $\{\,\cdot\,\}$ is uniformly bounded since it converges to $(b-a)^{2\alpha}\,\mathrm{B}(\alpha,\alpha)$. This completes the proof. ■

6.1.4 Step 3: Proof of the exponential convergence

Now we are in the position to prove Theorem 3.6.

Proof. What is left here is to evaluate the term $\|\mathcal{V}u - \mathcal{V}_N^{\mathrm{SE}} u\|_{\mathbf{C}}$ in (6.27). From Theorem 3.3, we have $u \in \mathbf{H}^{\infty}(\psi^{\mathrm{SE}}(\mathscr{D}_d))$. Hence $k(t,\cdot)u(\cdot)Q(\cdot) \in \mathbf{L}_{\alpha}(\psi^{\mathrm{SE}}(\mathscr{D}_d))$, and we can apply Theorem 2.3. This gives the desired estimate. ■

6.2 Convergence analysis of the DE-Sinc-Nyström methods

Next we prove the DE case, i.e., Theorem 3.7.

6.2.1 Sketch of the proof

The proof is similar to the SE case. Let us define an operator $\mathcal{V}_N^{\mathrm{DE}}$ as

$$\mathcal{V}_N^{\mathrm{DE}}[f](t) = \sum_{j=-N}^{N} k(t, \psi^{\mathrm{DE}}(jh)) f(\psi^{\mathrm{DE}}(jh)) \{\psi^{\mathrm{DE}}\}'(jh) J(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(t)), \tag{6.29}$$

which is the approximation of $\mathcal{V}f$ by the DE-Sinc indefinite integration. Then consider the following three equations:

$$(\mathcal{I} - \mathcal{V})u = g, \tag{6.30}$$

$$(\mathcal{I} - \mathcal{V}_N^{\mathrm{DE}})v = g, \tag{6.31}$$

$$(I_m - V_m^{\mathrm{DE}})\boldsymbol{c}_m = \boldsymbol{g}_m^{\mathrm{DE}}. \tag{6.32}$$

The first equation is nothing but (1.1), second is (3.13), and third is (3.16) itself. In order to prove the existence of $(I_m - V_m^{\mathrm{DE}})^{-1}$, which means feasibility of the scheme, we prove the following two claims:

1. the equation (6.32) is uniquely solvable if and only if the equation (6.31) is uniquely solvable (Lemma 6.8).
2. the equation (6.31) is uniquely solvable for all sufficiently large N (Lemma 6.10).

Combining Lemmas 6.8 and 6.10, we conclude Theorem 3.7.

6.2.2 Step 1: Equivalence of the solvability of (6.32) and (6.31)

The next lemma holds good, which can be proved in the same way as Lemma 6.1.

Lemma 6.8. The following statements are equivalent:

(A) The equation (6.31) has a unique solution $v \in \mathbf{C}$.

(B) The equation (6.32) has a unique solution $\boldsymbol{c}_m \in \mathbb{R}^m$.

6.2.3 Step 2: Solvability of the equation (6.31) for all sufficiently large N

Our task is to show that the conditions 1–4 in Theorem 6.2 are fulfilled with $\mathcal{X} = \mathcal{V}$ and $\mathcal{X}_n = \mathcal{V}_N^{\mathrm{DE}}$, under the assumptions in Theorem 3.7. The conditions 1–3 clearly hold by the same argument in the SE case. The condition 4 is confirmed by the following lemma.

Lemma 6.9. Let k satisfy the assumptions in Theorem 3.7. Then there exists a constant C independent of N such that

$$\|(\mathcal{V} - \mathcal{V}_N^{\mathrm{DE}})\mathcal{V}_N^{\mathrm{DE}}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})} \leq Ch^2, \tag{6.33}$$

where h is a mesh size defined by (2.17).

Proof. We show that there exists a constant C independent of N and f such that

$$\|(\mathcal{V} - \mathcal{V}_N^{\mathrm{DE}})\mathcal{V}_N^{\mathrm{DE}} f\|_{\mathbf{C}} \leq C\|f\|_{\mathbf{C}} h^2 \tag{6.34}$$

holds for all $f \in \mathbf{C}$. Let us define a function $F_i(t,s)$ as

$$F_i(t,s) = k(t,s)k(s,\psi^{\mathrm{DE}}(ih))Q(\psi^{\mathrm{DE}}(ih))J(i,h)(\{\psi^{\mathrm{DE}}\}^{-1}(t)). \tag{6.35}$$

Then $(\mathcal{V}-\mathcal{V}_N^{\mathrm{DE}})\mathcal{V}_N^{\mathrm{DE}} f$ can be rewritten as

$$(\mathcal{V}-\mathcal{V}_N^{\mathrm{DE}})\mathcal{V}_N^{\mathrm{DE}}[f](t) = \sum_{i=-N}^{N} \frac{f(\psi^{\mathrm{DE}}(ih))\pi\cosh(ih)}{b-a} \times \left\{ \int_a^t F_i(t,s)\,\mathrm{d}s - \sum_{j=-N}^{N} F_i(t,\psi^{\mathrm{DE}}(jh))\{\psi^{\mathrm{DE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(t)) \right\}. \tag{6.36}$$

First we evaluate the part in $\{\,\cdot\,\}$. From $k(t,\cdot)Q(\cdot)\in\mathbf{L}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$ and Lemma 6.4, it holds that

$$|F_i(t,z)Q(z)| \leq K|Q(z)|^\alpha K Q^\alpha(\psi^{\mathrm{DE}}(ih))\frac{5h}{\pi}\frac{\sinh(\pi d/h)}{\pi d/h} \tag{6.37}$$

for all integers i and $t\in[a,\,b]$ and $z\in\psi^{\mathrm{DE}}(\mathscr{D}_d)$. Therefore $F_i(t,\cdot)$ satisfies the assumptions of Theorem 2.4, from which we have

$$\begin{aligned}
&\left| \int_a^t F_i(t,s)\,\mathrm{d}s - \sum_{j=-N}^{N} F_i(t,\psi^{\mathrm{DE}}(jh))\{\psi^{\mathrm{DE}}\}'(jh)J(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(t)) \right| \\
&\leq \left\{ K^2 Q^\alpha(\psi^{\mathrm{DE}}(ih))\frac{5h}{\pi}\frac{\sinh(\pi d/h)}{\pi d/h} \right\}(b-a)^{2\alpha-1}C_{\alpha,d}^{\mathrm{DE}}\frac{\log(2dN/\alpha)}{N}\exp\left[\frac{-\pi dN}{\log(2dN/\alpha)}\right] \\
&= \frac{5K^2 Q^\alpha(\psi^{\mathrm{DE}}(ih))(b-a)^{2\alpha-1}C_{\alpha,d}^{\mathrm{DE}}}{\pi^2 d}\cdot h^3\left[\sinh(\pi d/h)\,\mathrm{e}^{-\pi d/h}\right].
\end{aligned} \tag{6.38}$$

The last equality holds from the formula (2.17). Furthermore, $|\sinh(\pi d/h)\,\mathrm{e}^{-\pi d/h}|\leq 1/2$ holds, which implies

$$\begin{aligned}
|(\mathcal{V}-\mathcal{V}_N^{\mathrm{DE}})\mathcal{V}_N^{\mathrm{DE}}[f](t)| &\leq \sum_{i=-N}^{N}\frac{|f(\psi^{\mathrm{DE}}(ih))|\pi\cosh(ih)}{b-a}\frac{5K^2 Q^\alpha(\psi^{\mathrm{DE}}(ih))(b-a)^{2\alpha-1}C_{\alpha,d}^{\mathrm{DE}}}{\pi^2 d}\cdot\frac{h^3}{2} \\
&\leq \|f\|_{\mathbf{C}}\frac{5K^2(b-a)^{2\alpha-2}C_{\alpha,d}^{\mathrm{DE}}}{2\pi^2 d}h^2\left\{ h\sum_{i=-N}^{N}\pi\cosh(ih)Q^\alpha(\psi^{\mathrm{DE}}(ih)) \right\}.
\end{aligned} \tag{6.39}$$

The term in $\{\,\cdot\,\}$ is uniformly bounded since it converges to $(b-a)^{2\alpha}\,\mathrm{B}(\alpha,\alpha)$. This shows the desired inequality (6.34). ■

Thus all of the conditions 1–4 in Theorem 6.2 are fulfilled, and the next lemma follows.

Lemma 6.10. Suppose that the assumptions in Theorem 3.7 are fulfilled. Then there exists a positive integer N_0 such that for all $N\geq N_0$ the equation (6.31) has a unique solution $v\in\mathbf{C}$. Furthermore there exists a constant C independent of N such that for all $N\geq N_0$

$$\|u-v\|_{\mathbf{C}} \leq C\|\mathcal{V}u-\mathcal{V}_N^{\mathrm{DE}}u\|_{\mathbf{C}}. \tag{6.40}$$

Proof. We have already confirmed the assumptions in Theorem 6.2, and what is left here is to show the boundedness of the term $\|(\mathcal{I}-\mathcal{V}_N^{\mathrm{DE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}$ (notice the inequality (6.12)). From the inequality (6.11), the claim follows if we prove that the term $\|\mathcal{V}_N^{\mathrm{DE}}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}$ is uniformly bounded. From Lemma 6.3 and $k(t,\cdot)Q(\cdot)\in\mathbf{L}_\alpha(\psi^{\mathrm{DE}}(\mathscr{D}_d))$, we have

$$\begin{aligned}
|\mathcal{V}_N^{\mathrm{DE}}[f](t)| &= \left| \sum_{j=-N}^{N} k(t,\psi^{\mathrm{DE}}(jh))f(\psi^{\mathrm{DE}}(jh))\frac{\pi\cosh(jh)Q(\psi^{\mathrm{DE}}(jh))}{b-a}J(j,h)(\{\psi^{\mathrm{DE}}\}^{-1}(t)) \right| \\
&\leq \frac{1.1K\|f\|_{\mathbf{C}}}{b-a}\left\{ h\sum_{j=-N}^{N}\pi\cosh(jh)Q^\alpha(\psi^{\mathrm{DE}}(jh)) \right\}
\end{aligned} \tag{6.41}$$

for all $f \in \mathbf{C}$. The term in $\{\,\cdot\,\}$ is uniformly bounded since it converges to $(b-a)^{2\alpha}\,\mathrm{B}(\alpha,\alpha)$. This completes the proof. ■

6.2.4 Step 3: Proof of the exponential convergence

Now we are in the position to prove Theorem 3.7.

Proof. It remains to evaluate the term $\|\mathcal{V}u - \mathcal{V}_N^{\mathrm{DE}}u\|_{\mathbf{C}}$ in (6.40). From Theorem 3.3, we have $u \in \mathbf{H}^{\infty}(\psi^{\mathrm{DE}}(\mathscr{D}_d))$. Hence $k(t,\cdot)u(\cdot)Q(\cdot) \in \mathbf{L}_{\alpha}(\psi^{\mathrm{DE}}(\mathscr{D}_d))$, and we can apply Theorem 2.4. Thus this theorem is established. ■

7 Proof of the main result 3

We prove only Theorem 3.11. Theorems 3.12–3.14 can be proved in exactly the same way.

Proof. If we show the following two inequalities:

$$\|I_m - V_m^{\mathrm{SE}}\|_{\infty} \leq \|\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}, \tag{7.1}$$

$$\|(I_m - V_m^{\mathrm{SE}})^{-1}\|_{\infty} \leq \|(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}, \tag{7.2}$$

then Theorem 3.11 is established. This is because, as proved in Lemma 6.7, $\|\mathcal{V}_N^{\mathrm{SE}}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}$ and $\|(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}$ are uniformly bounded.

We show (7.1) first. For a given arbitrary $\boldsymbol{c}_m = [c_{-N}, \ldots, c_N]^{\mathrm{T}}$, choose $\gamma \in \mathbf{C}$ with the properties

$$\|\gamma\|_{\mathbf{C}} = \|\boldsymbol{c}_m\|_{\infty} \quad \text{and} \quad \gamma(\psi^{\mathrm{SE}}(ih)) = c_i \quad (i = -N, \ldots, N). \tag{7.3}$$

Let us define a function f and a vector $\boldsymbol{f}_m^{\mathrm{SE}}$ as

$$f = (\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})\gamma, \quad \boldsymbol{f}_m^{\mathrm{SE}} = [f(\psi^{\mathrm{SE}}(-Nh)), \ldots, f(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}}. \tag{7.4}$$

Then we have

$$\begin{aligned}
\|(I_m - V_m^{\mathrm{SE}})\boldsymbol{c}_m\|_{\infty} &= \|\boldsymbol{f}_m^{\mathrm{SE}}\|_{\infty} \\
&\leq \|f\|_{\mathbf{C}} \\
&= \|(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})\gamma\|_{\mathbf{C}} \\
&\leq \|(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|\gamma\|_{\mathbf{C}} \\
&= \|(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|\boldsymbol{c}_m\|_{\infty},
\end{aligned} \tag{7.5}$$

which shows (7.1).

Next we show (7.2). Notice that $(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})^{-1}$ exists for all sufficiently large N from Lemma 6.7, and from Lemma 6.1 $(I_m - V_m^{\mathrm{SE}})^{-1}$ always exists if $(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})^{-1}$ exists. For a given arbitrary $\boldsymbol{c}_m = [c_{-N}, \ldots, c_N]^{\mathrm{T}}$, choose $\gamma \in \mathbf{C}$ with the properties (7.3). Furthermore define a function f and a vector $\boldsymbol{f}_m^{\mathrm{SE}}$ as

$$f = (\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})^{-1}\gamma, \quad \boldsymbol{f}_m^{\mathrm{SE}} = [f(\psi^{\mathrm{SE}}(-Nh)), \ldots, f(\psi^{\mathrm{SE}}(Nh))]^{\mathrm{T}}. \tag{7.6}$$

Then we have

$$\begin{aligned}
\|(I_m - V_m^{\mathrm{SE}})^{-1}\boldsymbol{c}_m\|_{\infty} &= \|\boldsymbol{f}_m^{\mathrm{SE}}\|_{\infty} \\
&\leq \|f\|_{\mathbf{C}} \\
&= \|(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})^{-1}\gamma\|_{\mathbf{C}} \\
&\leq \|(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|\gamma\|_{\mathbf{C}} \\
&= \|(\mathcal{I} - \mathcal{V}_N^{\mathrm{SE}})^{-1}\|_{\mathcal{L}(\mathbf{C},\mathbf{C})}\|\boldsymbol{c}_m\|_{\infty},
\end{aligned} \tag{7.7}$$

which shows (7.2). ■

8 Concluding remarks

In this paper, the Sinc-Nyström methods for (1.1) and (1.2) developed by Muhammad et al. [18] are considered, and the following three new theoretical results are given: 1) a way to estimate a tuning parameter d, 2) the exponential convergence of the schemes was rigorously proved, 3) the resulting system was proved to be well-conditioned. These results were established for both the SE- and DE-Sinc-Nyström methods. These results are also confirmed by the numerical experiments.

Based on the results in this paper, we can analyze the Sinc-collocation methods for Volterra integral equations [25], which will be reported somewhere else soon.

Acknowledgment

This work was supported by JSPS Grants-in-Aid for Scientific Research.

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