# THE INTERIOR TRANSMISSION PROBLEM FOR REGIONS WITH CAVITIES*

FIORALBA CAKONI †, DAVID COLTON ‡, AND HOUSSEM HADDAR§

**Abstract.** We consider the interior transmission problem in the case when the inhomogeneous medium has cavities, i.e. regions in which the index of refraction is the same as the host medium. In this case we establish the Fredholm property for this problem and show that transmission eigenvalues exist and form a discrete set. We also derive Faber-Krahn type inequalities for the transmission eigenvalues.

**Key words.** Inhomogeneous medium, interior transmission problem, inverse scattering, transmission eigenvalues.

**AMS subject classifications.** 35R30, 35Q60, 35J40, 78A25.

## 1. Introduction.

The interior transmission problem is a new class of boundary value problems for elliptic equations which was first discussed by Colton, Kirsch and Monk in the mid nineteen eighties in connection with the inverse scattering problem for acoustic waves in an inhomogeneous medium [6], [9]. Since that time the interior transmission problem has come to play a basic role in inverse scattering for both acoustic and electromagnetic waves (c. f. [2], [4], [10]). In its simplest formulation, the interior transmission problem is to determine under what conditions the coupled set of equations

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in } D \tag{1.1}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{1.2}$$

$$w - v = f \qquad \text{in } \partial D \tag{1.3}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = h \qquad \text{on } \partial D \tag{1.4}$$

has a unique solution $v, w$ where $D$ is a bounded simply connected domain with $C^2$ boundary $\partial D$, $\nu$ is the unit outward normal to $\partial D$, $f \in H^{\frac{3}{2}}(\partial D)$, $h \in H^{\frac{1}{2}}(\partial D)$ and $n \in L^\infty(D)$. Of particular interest is the *transmission eigenvalue problem*, i.e. the existence of values of $k > 0$ such that there exists a nontrivial solution to the homogeneous version of equations (1.1)-(1.4) (i.e. $f = h = 0$). Such values are called *transmission eigenvalues* (see Definition 3.3 below). Note that the case when $n(x) = 1$ for $x \in D$ is singular in the sense that in this case every $k > 0$ is an eigenvalue having infinite multiplicity. Hence until recently all of the results on the interior transmission problem have assumed that either $n(x) > 1$ for $x \in \overline{D}$ or $n(x) < 1$ for $x \in \overline{D}$. For a survey of known results for the interior transmission problem we refer the reader to the survey paper [7]. It is worth noting that the existence of transmission eigenvalues was only established very recently, again for case when either $n(x) > 1$ or $n(x) < 1$ for $x \in \overline{D}$ [13].

*The research of F.C. and D.C. was supported in part by the U.S. Air Force Office of Scientific Research under Grant FA 9550-08-1-0138.

†Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716-2553, USA (cakoni@math.udel.edu).

‡Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716-2553, USA (colton@math.udel.edu).

§INRIA Saclay Ile de France / CMAP Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France (haddar@cmapx.polytechnique.fr).

The case when $n(x)$ may be equal to one in a portion of $D$ was first considered in [1] in connection with problems in non-destructive testing. In particular, it was demonstrated in [1] that transmission eigenvalues can be determined from the far field pattern of the scattered wave and numerical evidence was presented suggesting that the presence of cavities in the scattering object, i.e. regions $D_0 \subset D$ for which $n(x) = 1$, cause the transmission eigenvalues to become larger. However, due to the singular nature of the transmission eigenvalue problem in this case, the theoretical results for the case when $n(x) > 1$ or $n(x) < 1$ for $x \in \overline{D}$ no longer apply. In particular, when cavities are present nothing is known about either the existence of a solution to the interior transmission problem or the existence and countability of transmission eigenvalue nor has an analog of the Faber-Krahn inequality given in [7, Theorem 7] been establish for transmission eigenvalue problems with cavities. The purpose of this paper is to provide these missing theoretical results. Due to the singular nature of the interior transmission problem with cavities, the mathematical methods we shall employ are different than those used in the non-singular case. In particular, it is no longer possible to write (1.1)-(1.4) as a fourth order elliptic equation for $v - w$ in $D$ which is the starting point for the analysis in the non-singular case [13], [14].

**2. Interior Transmission Problem.** We first consider the scattering problem of finding a function $u \in H^1_{loc}(\mathbb{R}^2)$ such that

$$\Delta u + k^2 n(x) u = 0 \qquad \text{in } \mathbb{R}^2 \tag{2.1}$$

$$u = u^s + u^i \tag{2.2}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{2.3}$$

where $x \in \mathbb{R}^2$, $r = |x|$, $k > 0$ is the wave number, $u^i(x) = e^{ikx\cdot d}$ with $|d| = 1$ is the incident field, $u^s$ is the scattered field satisfying the Sommerfeld radiation condition (2.3) uniformly in $\hat{x} = x/|x|$ and $n \in L^\infty(D)$ such that $\operatorname{Im} n(x) \geq 0$, Re $n(x) > 0$ for $x \in \overline{D}$ and $n(x) = 1$ for $x \in \mathbb{R}^2 \setminus \overline{D}$. Here $D$ is as defined in the Introduction. Then it can be shown [4] that $u^s$ has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x};\, d, k) + O(r^{-3/2}) \tag{2.4}$$

as $r \to \infty$ uniformly in $\hat{x}$ where $u_\infty$ is the *far field pattern* of the scattered field $u^s$ and we can define the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x};\, d, k) g(d)\, ds(d) \tag{2.5}$$

where $\Omega$ is the unit circle in $\mathbb{R}^2$. We note that by linearity $(Fg)(\hat{x})$ is the far field pattern corresponding to (2.1)-(2.3) where the incident field $e^{ikx\cdot d}$ is replaced by the *Herglotz wave function*

$$v_g(x) := \int_\Omega e^{ikx\cdot d} g(d) ds(d). \tag{2.6}$$

The following theorem is a reformulation and strengthening of Theorem 8.9 of [4]

THEOREM 2.1. *The far field operator is injective with dense range if and only if there does not exist a nontrivial solution $v, w \in L^2(D)$, $v - w \in H^2(D)$, of the* *transmission eigenvalue problem*

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in } D \tag{2.7}$$
$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{2.8}$$
$$w = v \qquad \text{in } \partial D \tag{2.9}$$
$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D \tag{2.10}$$

*such that $v$ is a Herglotz wave function.*

Values of $k > 0$ such that there exists a nontrivial solution of the homogeneous interior transmission problem (2.7)-(2.10) are called *transmission eigenvalues*. It can be easily shown that if $\operatorname{Im} n(x) > 0$ for $x$ in some neighborhood contained in $D$ then transmission eigenvalues do not exist [4]. On the other hand, if $n(x) > 1$ for $x \in \overline{D}$ (or $n(x) < 1$ for $x \in \overline{D}$) it is known that the Fredholm property holds for (1.1)-(1.4) and transmission eigenvalues exist and form a discrete set [13], [14]. In this paper we will allow the possibility of regions $D_0 \subset D$ such that $n(x) = 1$ in $D_0$ and in this case these results no longer apply. We will assume that $D_0$ is possibly multiply connected such that $D \setminus \overline{D}_0$ is connected and that $\partial D_0$ is a smooth curve. Our goal in the following is to establish the Fredholm property for (1.1)-(1.4) in this case, to show that transmission eigenvalues exist and form a discrete set and to establish Faber-Krahn type inequalities for transmission eigenvalues analogous to that of [7, Theorem 7] for the case where $D$ has no cavity. We remark that for the sake of presentation we limit ourselves to the two dimensional case. Everything in the following analysis holds true in the corresponding three dimensional case as well.

We now precisely formulate the interior transmission problem that is the main subject of this study. Let $D \subset \mathbb{R}^2$ be a simply connected and bounded region with $C^2$ boundary $\partial D$. Inside $D$ we consider a region $D_0 \subset D$ which can possibly be multiply connected such that $D \setminus \overline{D}_0$ is connected and assume that its boundary $\partial D_0$ is also a $C^2$ curve. Let $\nu$ denote the unit outward normal to $\partial D$ and $\partial D_0$ (see Fig 2.1).

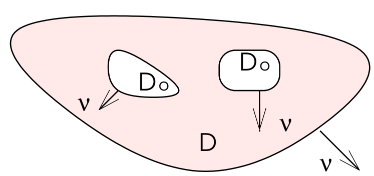


Fig. 2.1. *Configuration of the domain.*

Now let $n$ be an $L^\infty(D)$ complex valued function such that $n = 1$ in $D_0$ and $\operatorname{Re}(n) \geq c > 0$, $\operatorname{Im}(n) \geq 0$ almost everywhere in $D \setminus \overline{D}_0$. Given $f$ and $h$ we are interested in finding $v$ and $w$ that satisfy

$$\Delta w + k^2 n w = 0 \qquad \text{in } D \tag{2.11}$$
$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{2.12}$$
$$w - v = f \qquad \text{in } \partial D \tag{2.13}$$
$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = h \qquad \text{on } \partial D. \tag{2.14}$$

In the following, the interior transmission problem (2.11)-(2.14) is referred to as (ITP). Motivated by the case where $D_0 = \emptyset$, it is reasonable to define a weak solution to

(ITP) to be the pair of functions $w$ and $v$ in $L^2(D)$ that satisfy (2.11)-(2.12) in the distributional sense such that $u = w - v \in H^2(D)$ satisfies (2.13)-(2.14), where $f \in H^{3/2}(\partial D)$ and $h \in H^{1/2}(\partial D)$.

Assuming that $1/(n-1) \in L^\infty(D \setminus \overline{D}_0)$, let $w$ and $v$ be a weak solution to (ITP). Then $u = w - v$ satisfies

$$\Delta u + k^2 u = -k^2(n-1)w \qquad \text{in } D \setminus \overline{D}_0 \tag{2.15}$$

or

$$\Delta u + k^2 n u = -k^2(n-1)v \qquad \text{in } D \setminus \overline{D}_0. \tag{2.16}$$

Dividing both sides of (2.15) by $(n-1)$ and applying the operator $(\Delta + k^2 n)$ we obtain

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0 \qquad \text{in } D \setminus \overline{D}_0, \tag{2.17}$$

together with

$$u = f \quad \text{and} \quad \frac{\partial u}{\partial \nu} = h \quad \text{on } \partial D. \tag{2.18}$$

Inside $D_0$ one has

$$\left(\Delta + k^2\right) u = 0 \qquad \text{in } D_0, \tag{2.19}$$

with the continuity of the Cauchy data across $\partial D_0$

$$u^+ = u^- \qquad \text{and} \qquad \frac{\partial u^+}{\partial \nu} = \frac{\partial u^-}{\partial \nu}, \tag{2.20}$$

where, for a generic function $\phi$,

$$\phi^\pm(x) = \lim_{h \to 0^+} \phi(x \pm h\nu_x) \quad \text{and} \quad \frac{\partial \phi^\pm(x)}{\partial \nu_x} = \lim_{h \to 0^+} \nu_x \cdot \nabla \phi(x \pm h\nu_x) \tag{2.21}$$

for $x \in \partial D_0$ and $\nu$ the outward unit normal to $\partial D_0$ (see Fig 2.1).

The latter equations for $u$ are not sufficient to define $w$ and $v$ inside $\partial D_0$ and therefore one needs to add an additional unknown inside $D_0$, for instance the function $w$ that satisfies

$$\left(\Delta + k^2\right) w = 0 \qquad \text{in } D_0 \tag{2.22}$$

with the continuity of the Cauchy data across $\partial D_0$ that can be written using (2.15) as

$$\left(\frac{-1}{k^2(n-1)} \left(\Delta + k^2\right) u\right)^+ = w^- \;\text{ and }\; \frac{\partial}{\partial \nu} \left(\frac{-1}{k^2(n-1)} \left(\Delta + k^2\right) u\right)^+ = \frac{\partial w^-}{\partial \nu}. \tag{2.23}$$

We note that (2.23) is interpreted as equalities between functions in $H^{-1/2}(\partial D_0)$ and $H^{-3/2}(\partial D_0)$ respectively.

It is easily verified that the solutions $u \in H^2(D)$ and $w \in L^2(D_0)$ to (2.17)-(2.23) equivalently define a weak solution $w$ and $v$ to (2.11)-(2.14) by

$$w := \frac{-1}{k^2(n-1)} \left(\Delta + k^2\right) u \;\text{ in } D \setminus \overline{D}_0 \quad \text{and} \quad v := w - u \;\text{ in } D. \tag{2.24}$$

**3. The existence and uniqueness of a weak solution.** We shall establish existence and uniqueness results for the solution of (ITP) using a variational approach. The main difficulty in obtaining the variational formulation is to properly choose the function space that correctly handles the transmission conditions (2.20) and (2.23). More precisely, classical variational formulations of equations (2.17), (2.19) and (2.22) would require $u \in H^2(D \setminus \overline{D}_0) \cap H^1(D)$ and $v \in H^1(D_0)$ but this regularity is not sufficient to variationally treat all boundary terms in (2.20) and (2.23). The proposed variational space in the following treats equation (2.17) variationally and includes (2.19)-(2.20) into the variational space. More precisely we define

$$V(D, D_0, k) := \{u \in H^2(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0\} \tag{3.1}$$

which is a Hilbert space equipped with the $H^2(D)$ scalar product and look for the solution $u$ in $V(D, D_0, k)$. We also consider the closed subspace

$$V_0(D, D_0, k) := \{u \in H_0^2(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0\} \tag{3.2}$$

where

$$H_0^2(D) := \{u \in H^2(D) \text{ such that } u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \;\text{ on } \partial D\}. \tag{3.3}$$

Let $u \in V(D, D_0, k)$ and consider a test function $\psi \in V_0(D, D_0, k)$. For the sake of presentation we assume that $u$ and $\psi$ are regular enough to justify the various integration by parts and then use a denseness argument. Multiplying (2.17) by $\psi$ and integrating by parts we obtain

$$\begin{aligned} 0 &= \int_{D\setminus\overline{D}_0} (\Delta + k^2 n) \frac{1}{n-1} \left(\Delta + k^2\right) u \,\bar{\psi}\, dx \qquad (3.4) \\ &= \int_{D\setminus\overline{D}_0} \left( \left(\Delta + k^2\right) \frac{1}{n-1} \left(\Delta + k^2\right) u + k^2 (\Delta + k^2) u \right) \bar{\psi}\, dx \\ &= \int_{D\setminus\overline{D}_0} \frac{1}{n-1} \left(\Delta + k^2\right) u \, \left(\Delta + k^2\right) \bar{\psi}\, dx + k^4 \int_{D\setminus\overline{D}_0} u\, \bar{\psi}\, dx + k^2 \int_{D\setminus\overline{D}_0} \Delta u\, \bar{\psi}\, dx \\ &+ \int_{\partial D_0} \frac{1}{n-1} \left(\Delta + k^2\right) u \frac{\partial \bar{\psi}}{\partial \nu}\, ds - \int_{\partial D_0} \frac{\partial}{\partial \nu} \left( \frac{1}{n-1} \left(\Delta + k^2\right) u \right) \bar{\psi}\, ds. \end{aligned}$$

Using the fact that $\bar{\psi} \in V_0(D, D_0, k)$, the boundary conditions (2.23) and equation (2.22) we obtain that

$$\int_{\partial D_0} \frac{1}{n-1} \left(\Delta + k^2\right) u \frac{\partial \bar{\psi}}{\partial \nu}\, ds - \int_{\partial D_0} \frac{\partial}{\partial \nu} \left( \frac{1}{n-1} \left(\Delta + k^2\right) u \right) \bar{\psi}\, ds = 0. \tag{3.5}$$

Therefore we finally have that

$$\int_{D\setminus\overline{D}_0} \frac{1}{n-1} \left(\Delta + k^2\right) u \, \left(\Delta + k^2\right) \bar{\psi}\, dx + k^2 \int_{D\setminus\overline{D}_0} (\Delta u + k^2 u)\, \bar{\psi}\, dx = 0, \tag{3.6}$$

which is required to be valid for all $\psi \in V_0(D, D_0, k)$.

For given $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$ let $\theta \in H^2(D)$ be the lifting function [11] such that $\theta = f$ and $\partial\theta/\partial\nu = h$ on $\partial D$ and $\|\theta\|_{H^2(D)} \leq c \left( \|f\|_{H^{\frac{3}{2}}(\partial D)} + \|h\|_{H^{\frac{1}{2}}(\partial D)} \right)$ for some $c > 0$. Using a cutoff function we can guarantee that $\theta = 0$ in $D_\theta$ such that $D_0 \subset D_\theta \subset D$. The variational formulation amounts to finding $u_0 = u - \theta \in V_0(D, D_0, k)$ such that

$$\begin{aligned} &\int_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(\Delta + k^2\right) u_0 \left(\Delta + k^2\right) \bar{\psi}\, dx + k^2 \int_{D\setminus\overline{D}_0} (\Delta u_0 + k^2 u_0)\, \bar{\psi}\, dx \\ &= \int_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(\Delta + k^2\right) \theta \left(\Delta + k^2\right) \bar{\psi}\, dx + k^2 \int_{D\setminus\overline{D}_0} (\Delta \theta + k^2 \theta)\, \bar{\psi}\, dx \end{aligned} \tag{3.7}$$

for all $\psi \in V_0(D, D_0, k)$.

As one can see, the above variational formulation involves only $u$ (in particular it does not involve $w$). The following lemma shows that the existence of $w$ is implicitely contained in the variational formulation.

Lemma 3.1. *Assume that $k^2$ is not both a Dirichlet and a Neumann eigenvalue for $-\Delta$ in $D_0$, and let $(\beta, \alpha) \in H^{-\frac{1}{2}}(\partial D_0) \times H^{-\frac{3}{2}}(\partial D_0)$ such that*

$$\langle \beta,\, \partial\psi/\partial\nu \rangle_{H^{-\frac{1}{2}}(\partial D_0), H^{\frac{1}{2}}(\partial D_0)} - \langle \alpha,\, \psi \rangle_{H^{-\frac{3}{2}}(\partial D_0), H^{\frac{3}{2}}(\partial D_0)} = 0 \tag{3.8}$$

*for all $\psi \in V_0(D, D_0, k)$. Then there exists a unique $w \in L^2(D_0)$ such that $\Delta w + k^2 w = 0$ in $D_0$ and $(w, \partial w/\partial\nu) = (\beta, \alpha)$ on $\partial D_0$.*

*Proof.* Assume that $k^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $D_0$. Let $w \in L^2(D_0)$ be a weak solution of $\Delta w + k^2 w = 0$ in $D_0$ and $w = \beta$ on $\partial D_0$ (see Remark below on how one can construct this solution from $H^1(D_0)$ solutions by using a classical duality argument, (i.e. the traces of $w$ and $\partial w/\partial\nu$ can be defined in this case by duality argument, see also [11]). Then applying Green's formula between $w$ and a test function $\psi \in V_0(D, D_0, k)$ we get

$$\langle w,\, \partial\psi/\partial\nu \rangle_{H^{-\frac{1}{2}}(\partial D_0), H^{\frac{1}{2}}(\partial D_0)} - \langle \partial w/\partial\nu,\, \psi \rangle_{H^{-\frac{3}{2}}(\partial D_0), H^{\frac{3}{2}}(\partial D_0)} = 0 \tag{3.9}$$

and therefore

$$\langle \partial w/\partial\nu - \alpha,\, \psi \rangle_{H^{-\frac{3}{2}}(\partial D_0), H^{\frac{3}{2}}(\partial D_0)} = 0 \tag{3.10}$$

for all $\psi \in V_0(D, D_0, k)$. We know that the traces of Herglotz wave functions are dense in $H^{\frac{3}{2}}(\partial D_0)$ (see [15, Theorem 4]) provided that $k^2$ is not a Dirichlet eigenvalue for $-\Delta$ in $D_0$ and, since $V_0(D, D_0, k)$ contains the set of Herglotz wave functions, we can conclude that the traces on $\partial D_0$ of functions in $V_0(D, D_0, k)$ are dense in $H^{\frac{3}{2}}(\partial D_0)$. Hence $\partial w/\partial\nu = g$ and the result follows. The case when $k^2$ is a not a Neumann eigenvalue can be treated by choosing $w \in L^2(D_0)$ to be a weak solution of $\Delta w + k^2 w = 0$ in $D_0$ such that $\partial w/\partial\nu = \alpha$ on $\partial D_0$ and using the densness of normal traces on $\partial D_0$ of functions in $V_0(D, D_0, k)$ in $H^{\frac{1}{2}}(\partial D_0)$ (the densness result follows from [15, Theorem 3]). The uniqueness of $w$ is obvious. ∎

Remark 1. We briefly recall the construction of $L^2$ solutions for the Helmholtz equation in $D_0$. Assume that $k^2$ is not a Dirichlet eigenvalue and let $g \in H^{\frac{1}{2}}(\partial D_0)$ and $u \in H^1(D_0)$ satisfy $\Delta u + k^2 u = 0$ in $D_0$ and $u = g$ on $\partial D_0$. Let $v \in H^1(D_0)$ to be solution of $\Delta v + k^2 v = u$ such that $v = 0$ on $\partial D_0$. Then standard regularity results imply that $v \in H^2(D_0)$ and there exists a constant $c$ independent of $v$ and $u$ such that $\|v\|_{H^2(D_0)} \le c\|u\|_{L^2(D_0)}$. Using Green's formula one easily obtains

$$\begin{aligned} \|u\|^2_{L^2(D_0)} = \left| \int_{D_0} g\, \partial v/\partial\nu \right| &\le \|g\|_{H^{-\frac{1}{2}}(\partial D_0)} \|\partial v/\partial\nu\|_{H^{\frac{1}{2}}(\partial D_0)} \\ &\le C\|g\|_{H^{-\frac{1}{2}}(\partial D_0)} \|u\|_{L^2(D_0)} \end{aligned} \tag{3.11}$$

and therefore the solution operator $g \to u$ is continuous from $H^{-\frac{1}{2}}(\partial D_0)$ into $L^2(D_0)$. Similar arguments also show that if $k^2$ is not an eigenvalue for the Neumann problem then the solution operator $g \to u$ where $u \in H^1(D_0)$ satisfying $\Delta u + k^2 u = 0$ in $D_0$ and $\partial u/\partial \nu = g$ is continous from $H^{-\frac{3}{2}}(\partial D_0)$ into $L^2(D_0)$.

Remark 2. If the solution of the variational problem (3.7) is in $H^4(D \setminus \overline{D}_0)$ then one can use the Calderòn projection operator to construct $w$ in $D_0$ and thus avoid the assumption on $k^2$ in Lemma 3.1.

We now can state the equivalence between weak solutions to (ITP) and solutions to the variational formulation (3.7).

Theorem 3.2. *Assume that $k^2$ is not both a Dirichlet and a Neumann eigenvalue for $-\Delta$ in $D_0$ and that $1/(n-1) \in L^\infty(D \setminus \overline{D}_0)$. Then the existence and uniqueness of a weak solution $w$ and $v$ to the interior transmission problem (2.11)-(2.14) is equivalent to the existence and uniqueness of a solution $u_0$ of the variational problem (3.7).*

*Proof.* It remains only to verify that any solution to (3.7) defines a weak solution $w$ and $v$ to the the interior transmission problem (2.11)-(2.14). Taking a test function $\psi$ to be a $C^\infty$ function with compact support in $D \setminus \overline{D}_0$ one can easily verify from (3.6) that $u$ satisfies (2.17). In particular, the function

$$w^+ := \left( -\frac{1}{k^2(n-1)} (\Delta + k^2) u \right)_{|D \setminus \overline{D}_0}$$

satisfies $w^+ \in L^2(D \setminus \overline{D}_0)$ and $(\Delta + k^2 n) w^+ = 0$ in $D \setminus \overline{D}_0$. For an arbitrary test function $\psi \in C^\infty(D \setminus \overline{D}_0)$ we can apply Green's formula and (3.6) to obtain

$$\left\langle w^+, \partial\psi/\partial\nu \right\rangle_{H^{-\frac{1}{2}}(\partial D_0), H^{\frac{1}{2}}(\partial D_0)} - \left\langle \partial w^+/\partial\nu, \psi \right\rangle_{H^{-\frac{3}{2}}(\partial D_0), H^{\frac{3}{2}}(\partial D_0)} = 0. \tag{3.12}$$

Finally, applying Lemma 3.1, we now obtain the existence of $w^- \in L^2(D_0)$ satisfying (2.22) and (2.23). □

We now proceed with the proof of existence and uniqueness of a variational solution. In the following we exclude the values of $k$ for which the uniqueness does not hold, namely the so-called transmission eigenvalues.

Definition 3.3. *Values of $k > 0$ for which the homogeneous variational problem (3.7) (i.e. (3.7) for $\theta = 0$) has nontrivial solutions $u_0$ are called transmission eigenvalues.*

Remark 3. Note that by Theorem 3.2, if $k^2$ is not both a Dirichlet and a Neumann eigenvalue, then if $k$ is a transmission eigenvalue there exists a nontrivial weak solution to (2.11)-(2.14) for $f = h = 0$ (see also Remark 2 above).

Theorem 3.4. *Let $f \in H^{\frac{3}{2}}(\partial D)$ and $h \in H^{\frac{1}{2}}(\partial D)$ and assume that $n \in L^\infty(D)$ is such that $n = 1$ in $D_0$, $Re(n) \geq c > 0$ and $Im(n) \geq 0$ almost everywhere in $D \setminus \overline{D}_0$. Assume further that either $Re(n-1) \geq 1/\gamma > 0$ or $Re(1-n) \geq 1/\gamma > 0$ almost everywhere in $D \setminus \overline{D}_0$ for some constant $\gamma$. Then the interior transmission problem (3.7) has a unique solution provided that $k$ is not a transmission eigenvalue. This solution depends continuously on the data $f$ and $h$.*

*Proof.* Let us define the following bounded sesquilinear forms on $V_0(D, D_0, k) \times V_0(D, D_0, k)$:

$$\begin{aligned} \mathcal{A}(u_0, \psi) \quad = \quad & \pm \int_{D \setminus \overline{D}_0} \frac{1}{n-1} \left( \Delta u_0 \, \Delta \bar{\psi} + \nabla u_0 \cdot \nabla \bar{\psi} + u_0 \, \bar{\psi} \right) \, dx \\ & + \int_{D_0} \left( \nabla u_0 \cdot \nabla \bar{\psi} + u_0 \, \bar{\psi} \right) \, dx \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\mathcal{B}_k(u_0,\psi) &= \pm k^2\int_{D\setminus\overline{D}_0}\frac{1}{n-1}\left(u_0(\Delta\bar{\psi}+k^2\bar{\psi})+(\Delta u_0+k^2nu_0)\bar{\psi}\right)dx\\ &\mp\int_{D\setminus\overline{D}_0}\frac{1}{n-1}\left(\nabla u_0\cdot\nabla\bar{\psi}+u_0\,\bar{\psi}\right)\,dx\\ &-\int_{D_0}\left(\nabla u_0\cdot\nabla\bar{\psi}+u_0\,\bar{\psi}\right)\,dx\end{aligned}\tag{3.14}$$

where the upper sign corresponds to the case when $\mathrm{Re}(n)>1$ whereas the lower sign to the case when $\mathrm{Re}(n)<1$. In terms of these forms the variational equation (3.7) for $u_0\in V_0(D,D_0,k)$ becomes

$$\mathcal{A}(u_0,\psi)+\mathcal{B}_k(u_0,\psi)=\mathcal{A}(\theta,\psi)+\mathcal{B}_k(\theta,\psi)\quad\text{for all}\ \ \psi\in V_0(D,D_0,k).\tag{3.15}$$

It is clear that if the real part of $1/(n-1)$ is positive definite or negative definite then there exists a positive constant $\gamma$, that only depends on $n$, such that

$$\mathcal{A}(u_0,u_0)\geq\gamma(\|\Delta u\|^2_{L^2(D\setminus\overline{D}_0)}+\|u\|^2_{H^1(D)}).\tag{3.16}$$

Let $\epsilon=1/(1+k^4)$, so that $0<\epsilon<1$ and $\epsilon k^4<1$. Since $\Delta u_0=-k^2u_0$ in $D_0$ one also has that

$$\begin{aligned}\mathcal{A}(u_0,u_0) &\geq \gamma\epsilon\|\Delta u\|^2_{L^2(D)}+\gamma(1-\epsilon k^4)\|u\|^2_{H^1(D)}\\ &= (\gamma/(1+k^4))(\|\Delta u\|^2_{L^2(D)}+\|u\|^2_{H^1(D)}).\end{aligned}\tag{3.17}$$

From standard elliptic regularity results we deduce that

$$\mathcal{A}(u_0,u_0)\geq(\tilde{\gamma}/(1+k^4))\|u_0\|^2_{H^2(D)},\tag{3.18}$$

where $\tilde{\gamma}$ only depends on $D$ and $n$. Therefore $\mathcal{A}$ defines a continuous and positive definite sesquilinear form on $V_0(D,D_0,k)\times V_0(D,D_0,k)$. Moreover if $|1/(n-1)|$ and $n$ are bounded then the compact embedding of $H^2_0(D)$ into $H^1(D)$ (Rellich's theorem) implies that $\mathcal{B}_k$ defines a compact perturbation of $\mathcal{A}$ while the right hand side of (3.15) defines a continuous antilinear form on $V_0(D,D_0,k)$. The result of our theorem now follows from an application of the Fredholm alternative. □

**4. Transmission eigenvalues.** Now we turn our attention to the study of the homogeneous interior transmission problem and transmission eigenvalues as defined in Definition 3.3. The assumptions on $D$, $D_0$ and the index of refraction $n$ are those stated in Section 2. In particular, throughout this section we assume that $n\in L^\infty(D)$ such that $n=1$ in $D_0$ and $\mathrm{Re}(n)\geq c>0$, almost everywhere in $D\setminus\overline{D}_0$. In addition we assume that either $\mathrm{Re}(n-1)\geq\alpha>0$ or $\mathrm{Re}(1-n)\geq\alpha>0$ almost everywhere in $D\setminus\overline{D}_0$ for some constant $\alpha$. We first note that $k$ is a transmission eigenvalue if and only if the homogeneous problem

$$\mathcal{A}(u_0,\psi)+\mathcal{B}_k(u_0,\psi)=0\text{ for all }\psi\in V_0(D,D_0,k)\tag{4.1}$$

has a nontrivial solution $u_0\in V_0(D,D_0,k)$. Taking $\psi=u_0$ we obtain

$$\begin{aligned}0=&\int_{D\setminus\overline{D}_0}\frac{1}{n-1}|\Delta u_0+k^2u_0|^2\,dx+k^4\int_{D\setminus\overline{D}_0}|u_0|^2\,dx\\ &-k^2\int_{D\setminus\overline{D}_0}|\nabla u_0|^2\,dx-k^2\int_{\partial D_0}\bar{u}_0^+\,\frac{\partial u_0^+}{\partial\nu}\,ds.\end{aligned}\tag{4.2}$$

In order to study transmission eigenvalues it suffices to study (4.2).

We now proceed with the proof of the fact that if the index of refraction $n$ has a nonzero imaginary part in $D \setminus \overline{D}_0$ then there are no transmission eigenvalues.

THEOREM 4.1. *If $n \in L^\infty(D)$ is such that $\mathrm{Im}(n) > 0$ almost everywhere in $D \setminus \overline{D}_0$, then there are no transmission eigenvalues.*

*Proof.* Using Green's first identity for $u_0$ in $D_0$ and the continuity of the Cauchy data of $u_0$ across $\partial D_0$ we can re-write (4.2) as

$$\begin{aligned} 0 = &\int_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta u_0 + k^2 u_0|^2\,dx + k^4\int_{D\setminus\overline{D}_0}|u_0|^2\,dx - k^2\int_{D\setminus\overline{D}_0}|\nabla u_0|^2\,dx \\ &+ k^4\int_{D_0}|u_0|^2\,dx - k^2\int_{D_0}|\nabla u_0|^2\,dx. \end{aligned} \tag{4.3}$$

Since $\mathrm{Im}(1/(n-1)) < 0$ in $D \setminus \overline{D}_0$ and all the terms in the above equation are real except for the first one, by taking the imaginary part we obtain that $\Delta u_0 + k^2 u_0 = 0$ in $D \setminus \overline{D}_0$ and since $u_0$ has zero Cauchy data zero on $\partial D$ we obtain that $u_0 = 0$ in $D \setminus \overline{D}_0$ and therefore $k$ is not a transmission eigenvalue. Note that the proof requires that $\mathrm{Im}(n) > 0$ a.e. in all of $D \setminus \overline{D}_0$. ∎

From now on we assume that $\mathrm{Im}(n) = 0$ and set $n_* = \inf_{D\setminus\overline{D}_0}(n)$ and $n^* = \sup_{D\setminus\overline{D}_0}(n)$ (essential infimum and supremum, respectively) .

**4.1. Faber-Krahn type inequalities for transmission eigenvalues.** We now want to show that if $k > 0$ is sufficiently small then $k$ is not a transmission eigenvalue. It suffices to show that if $u_0 \in V_0(D, D_0, k)$ satisfies (4.3) then $u_0$ is zero. To this end we first notice that since $u_0 \in H_0^2(D)$ we have that

$$\|u_0\|^2_{L^2(D)} \le \frac{1}{\lambda_1(D)}\|\nabla u_0\|^2_{L^2(D)} \tag{4.4}$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in $D$.

To obtain a lower bound for the first transmission eigenvalue we rewrite (4.2) in an equivalent form. In particular, combining terms differently, (4.2) takes the equivalent form

$$\begin{aligned} &\int_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta u_0 + k^2 n u_0|^2\,dx - k^4\int_{D\setminus\overline{D}_0} n|u_0|^2\,dx + k^2\int_{D\setminus\overline{D}_0}|\nabla u_0|^2\,dx \\ &\qquad -k^4\int_{D_0}|u_0|^2\,dx + k^2\int_{D_0}|\nabla u_0|^2\,dx = 0 \qquad u_0 \in V_0(D\setminus\overline{D}_0). \end{aligned} \tag{4.5}$$

Assume first that $1 + \alpha \le n_* \le n(x) \le n^*$ for $x \in D \setminus \overline{D}_0$ and some positive constant $\alpha > 0$. In this case $1/(n-1) > 0$ almost everywhere in $D \setminus \overline{D}_0$ and therefore if the sum of the last four terms in (4.5) is nonnegative then obviously $k$ is not a transmission eigenvalue. Hence we have

$$\begin{aligned} -k^2&\int_{D\setminus\overline{D}_0} n|u_0|^2\,dx + \int_{D\setminus\overline{D}_0}|\nabla u_0|^2\,dx - k^2\int_{D_0}|u_0|^2\,dx + \int_{D_0}|\nabla u_0|^2\,dx \qquad (4.6)\\ &\ge \int_D |\nabla u_0|^2\,dx - k^2 n^*\int_D |u_0|^2\,dx \ge (\lambda_1(D) - k^2 n^*)\|u_0\|^2_{L^2(D)}. \end{aligned}$$

Therefore all $k > 0$ such that $k^2 \leq \dfrac{\lambda_1(D)}{n^*}$ are not transmission eigenvalues. This means that all transmission eigenvalues satisfy

$$k^2 > \frac{\lambda_1(D)}{n^*} \tag{4.7}$$

provided $n^* = \sup_{D\setminus\overline{D}_0} n > 1$.

Next, if $0 \leq n_* \leq n(x) \leq n^* < 1 - \beta$ for $x \in D \setminus \overline{D_0}$ and some positive constant $\beta > 0$ then $1/(1-n) > 0$ almost everywhere in $D \setminus \overline{D}_0$. Hence from (4.3) after multiplying by $-1$ we see that $k > 0$ is not a transmission eigenvalue as long as

$$\begin{aligned}-k^2\int_{D\setminus\overline{D}_0}|u_0|^2\,dx+\int_{D\setminus\overline{D}_0}|\nabla u_0|^2\,dx-k^2\int_{D_0}|u_0|^2\,dx+\int_{D_0}|\nabla u_0|^2\,dx&\\ \geq\int_D|\nabla u_0|^2\,dx-k^2\int_D|u_0|^2\,dx\geq(\lambda_1(D)-k^2)\|u_0\|^2_{L^2(D)}&\end{aligned} \tag{4.8}$$

which means $k^2 \leq \lambda_1(D)$. Hence all transmission eigenvalues satisfy

$$k^2 > \lambda_1(D) \tag{4.9}$$

provided $0 < n_* = \inf_{D\setminus\overline{D}_0} n < 1$.

For the case of $1 + \alpha \leq n_* \leq n(x) \leq n^*$ we can rewrite (4.7) in terms of the $L^p$-norm of $n$ for some $p > 1$. Let $\tilde{n}$ be equal to $n$ in $D \setminus \overline{D}_0$ and equal to 1 in $D_0$. Using Hölder's inequality we have

$$\int_D \tilde{n}|u_0|^2\,dx \leq \left(\int_D \tilde{n}^p\,dx\right)^{1/p}\left(\int_D |\tilde{u}_0|^{2p/(p-1)}\,dx\right)^{(p-1)/p} \tag{4.10}$$

where $\alpha = 2p/(p-1)$. Hence

$$\begin{aligned}&-k^2\int_{D\setminus\overline{D}_0}n|u_0|^2\,dx+\int_{D\setminus\overline{D}_0}|\nabla u_0|^2\,dx-k^2\int_{D_0}|u_0|^2\,dx+\int_{D_0}|\nabla u_0|^2\,dx\\ &\quad\geq\int_D|\nabla u_0|^2\,dx-k^2\int_D\tilde{n}|u_0|^2\,dx\\ &\quad\geq\left(\Lambda_p(D)-k^2\left(\int_{D\setminus\overline{D}_0}n^p\,dx+\text{area}(D_0)\right)^{1/p}\right)\left(\int_D|u_0|^{2p/(p-1)}\,dx\right)^{(p-1)/p}\end{aligned} \tag{4.11}$$

where

$$\Lambda_p(D) = \inf_{u_0\in H_0^1(D)} \frac{\int_D |\nabla u_0|^2\,dx}{\left(\int_D |u_0|^{2p/(p-1)}\,dx\right)^{(p-1)/p}} \tag{4.12}$$

which is shown to exist in [8, Theorem 9.2.8]. Therefore, all transmission eigenvalues satisy

$$k^2 > \frac{\Lambda_p(D)}{\left(\int_{D\setminus\overline{D}_0} n^p\,dx + \text{area}(D_0)\right)^{1/p}}, \qquad p > 1. \tag{4.13}$$

Finally, again for the case of $1+\alpha \le n_* \le n(x) \le n^*$, we can refine (4.7) in order to involve the geometry of $D_0$. To this end we introduce

$$\lambda(D, D_0, k) := \inf_{\psi \in V(D,D_0,k), \psi \neq 0} \|\nabla \psi\|^2_{L^2(D)} / \|\psi\|^2_{L^2(D\setminus \overline{D}_0)}. \tag{4.14}$$

We remark that for all $k \ge 0$, we have that $\lambda_1(D) \le \lambda(D, D_0, k) \le \lambda_1(D \setminus \overline{D}_0)$ and moreover $\lim_{|D_0|\to 0} \lambda(D, D_0, k) = \lambda_1(D)$. Let

$$\lambda(D, D_0) = \inf_{k \ge 0} \lambda(D, D_0, k). \tag{4.15}$$

Then $\lambda_1(D) \le \lambda(D, D_0, k) \le \lambda_1(D \setminus \overline{D}_0)$ and $\lim_{|D_0|\to 0} \lambda_1(D, D_0) = \infty$. We similarly introduce

$$\mu(D, D_0, k) = \inf_{u_0 \in V_0(D,D_0,k), u_0 \neq 0} \|\nabla u_0\|^2_{L^2(D)} / \|u_0\|^2_{L^2(D_0)}. \tag{4.16}$$

It is clear that for all $k \ge 0$, $\mu(D, D_0, k) > \lambda_1(D)$ and $\lim_{|D_0|\to 0} \mu(D, D_0, k) = \infty$. Next setting

$$\mu(D, D_0) = \inf_{k \ge 0} \mu(D, D_0, k) \tag{4.17}$$

we have that $\mu(D, D_0) \ge \lambda_1(D)$ and $\lim_{|D_0|\to 0} \mu(D, D_0) = \infty$. Hence from the definition of $\mu(D, D_0)$ and $\lambda(D, D_0)$ it is easy to see that for any $\theta$ between 0 and 1,

$$\begin{aligned} &-k^2 \int_{D\setminus \overline{D}_0} n|u_0|^2 \, dx + \int_{D\setminus \overline{D}_0} |\nabla u_0|^2 \, dx - k^2 \int_{D_0} |u_0|^2 \, dx + \int_{D_0} |\nabla u_0|^2 \, dx \\ &\ge (\theta \lambda(D, D_0) - k^2 n^*) \|u_0\|^2_{L^2(D\setminus \overline{D}_0)} + ((1-\theta)\mu(D, D_0) - k^2) \|u_0\|^2_{L^2(D_0)}. \end{aligned} \tag{4.18}$$

Setting

$$\begin{aligned} k_1^2(D, D_0, n^*) :&= \max_{0 \le \theta \le 1} (\min(\theta \lambda(D, D_0)/n^*, (1-\theta)\mu(D, D_0))) \\ &= \frac{\lambda(D, D_0)\mu(D, D_0)}{\lambda(D, D_0) + n^* \mu(D, D_0)} \end{aligned} \tag{4.19}$$

we have that all transmission eigenvalues satisfy $k^2 > k_1^2(D, D_0, n^*)$. Note that if $D_0 \neq \emptyset$ then $k_1^2(D, D_0, n^*) > \lambda_1(D)/n^*$ and $k_1^2(D, D_0, n^*) \to \lambda_1(D)/n^*$ as $|D_0| \to 0$.

### 4.2. The existence and discreteness of transmission eigenvalues.

**4.2.1. Discreteness of transmission eigenvalues.** First we show that the transmission eigenvalues form at most a discrete set. To this end, with the aim of using the analytic Fredholm theory, we first show that an operator associated with the resolution of the ITP problem is analytic with respect to $k \in C$ in some neighborhood of the real axis.

As indicated in the previous sections, finding transmission eigenvalues is equivalent to finding $k > 0$ such that the problem

$$\mathcal{A}(u_0, \psi) + \mathcal{B}_k(u_0, \psi) = 0 \text{ for all } \psi \in V(D, D_0, k) \tag{4.20}$$

has non trivial solutions $u_0 \in V(D, D_0, k)$. This is equivalent to finding the values of $k$ for which

$$A_k + B_k : V(D, D_0, k) \to V(D, D_0, k) \tag{4.21}$$

has a nontrivial kernel where $A_k$ and $B_k$ denote the operators associated with $\mathcal{A}$ and $\mathcal{B}_k$, respectively, by using the Riesz representation theorem. According to last section the operator $A_k$ is invertible and the operator $B_k$ is compact.

To avoid dealing with function spaces depending on $k$ we shall make use of an (analytic) operator $\tilde{P}_k$ from $H_0^2(D)$ into $V(D, D_0, k)$ (that mimics the effects of a projection operator) in order to build (an analytic) extension of $A_k$ and $B_k$ with operators acting on $H_0^2(D)$.

Let $k$ be complex with positive real part. For $u \in H_0^2(D)$ we define $\theta_k(u)$ by

$$(\theta_k u)(x) = \frac{1}{4} \int_{D_0} (\Delta u + k^2 u)(y) Y_0(k|x-y|)\, dy \tag{4.22}$$

where $Y_0$ denotes the Bessel function of second kind of order zero. Then, using standard continuity properties of volume potentials, $\theta_k u \in H^2_{loc}(\mathbb{R}^2)$ and there exists a constant $C(k)$ such that

$$\|\theta_k u\|_{H^2(D)} \leq C(k) \|\Delta u + k^2 u\|_{L^2(D_0)}. \tag{4.23}$$

We recall that $Y_0$ is analytic outside the non-positive real axis. We also observe that $t \to Y_0(kt) - 2/\pi \log(t)$ is a smooth function for real positive $t$ and all $k \in \mathbb{C}$ with $\mathrm{Re}(k) > 0$. We then conclude that $\theta_k : H_0^2(D) \to H_0^2(D)$ depends analytically with respect to $k \in \mathbb{C}$ with $\mathrm{Re}(k) > 0$.

Let $\chi$ be a $C^\infty$ cutoff function that equals 1 in $D_0$ and 0 outside $D$. Then we define the continuous operator $\tilde{P}_k : H_0^2(D) \to H_0^2(D)$ by

$$\tilde{P}_k u = u - \chi \theta_k u. \tag{4.24}$$

We first observe that

$$\theta_k u = 0 \quad \text{and} \quad \tilde{P}_k u = u \quad \forall\, u \in V(D, D_0, k). \tag{4.25}$$

Furthermore, since

$$\Delta \theta_k u + k^2 \theta_k u = \Delta u + k^2 u \quad \text{in } D_0, \tag{4.26}$$

we also have that $\tilde{P}_k u \in V(D, D_0, k)$ for every $u \in H_0^2(D)$. We finally observe that, by analyticity of $\theta_k$, $\tilde{P}_k$ also depends analytically on complex $k$ with positive real part.

Using the Riesz representation theorem, we now introduce the operators $\tilde{A}_k$ and $\tilde{B}_k$ from $H_0^2(D)$ into $H_0^2(D)$ respectively defined by

$$\begin{aligned} (\tilde{A}_k u, v)_{H^2(D)} &= \mathcal{A}(\tilde{P}_k u, \overline{\tilde{P}_k \overline{v}}) + \alpha(\theta_k u, \overline{\theta_k \overline{v}})_{H^2(D)} \\ (\tilde{B}_k u, v)_{H^2(D)} &= \mathcal{B}_k(\tilde{P}_k u, \overline{\tilde{P}_k \overline{v}}) \end{aligned} \tag{4.27}$$

for all $u$ and $v$ in $H_0^2(D)$ where $\alpha$ is a sufficiently large positive constant that will be fixed later (and is independent of $k$). The analyticity of $P_k$ and $\theta_k$ as well as the expression for $\mathcal{B}_k$ show that $\tilde{A}_k$ and $\tilde{B}_k$ depend analytically on $k \in \mathbb{C}$ with $\mathrm{Re}(k) > 0$. Moreover, the operator $\tilde{B}_k$ is compact.

Observe that, if $k$ is real, then for $v \in V(D, D_0, k)$ we have that $\overline{v} \in V(D, D_0, k)$, and hence from (4.25), we have that

$$\tilde{A}_k u = A_k u \quad \tilde{B}_k u = B_k u \quad \forall\, u \in V(D, D_0, k) \text{ and } \forall\, k \in \mathbb{R} \tag{4.28}$$

Hence we conclude that for real $k$ if $A_k + B_k$ is not injective then $\tilde{A}_k + \tilde{B}_k$ is not injective. Consequently, in order to show that the set of transmission eigenvalues is at most discrete, it is sufficient to prove that the set $k$ for which $\tilde{A}_k + \tilde{B}_k$ is not injective is at most discrete. For that purpose we shall prove the following lemma:

LEMMA 4.2. *Assume that $n$ satisfies the assumptions of Theorem 3.4. Let $k$ be positive and real and let $\tilde{A}_k$ and $\tilde{B}_k$ be the operators defined by (4.27). Then*

- *There exist $\alpha_0$ independent of $k$ such that for all $\alpha \geq \alpha_0$ the operator $\tilde{A}_k$ is strictly coercive for all $k > 0$.*
- *There exist $k_0$ such that for all $0 < k \leq k_0$ the operator $\tilde{A}_k + \tilde{B}_k$ is injective.*

*Proof.* Assume that $k$ is real and consider the first statement. According to the definition of $\theta_k$ and $\tilde{P}_k$ we have that $\theta_k(\bar{u}) = \overline{\theta_k(u)}$ and $\tilde{P}_k(\bar{u}) = \overline{\tilde{P}_k(u)}$. Therefore

$$(\tilde{A}_k u, u)_{H^2(D)} = \mathcal{A}(\tilde{P}_k u, \tilde{P}_k u) + \alpha \|\theta_k u\|^2_{H^2(D)}. \tag{4.29}$$

From the coercivity of $\mathcal{A}$ on $V(D, D_0, k)$ we have that

$$(\tilde{A}_k u, u)_{H^2(D)} \geq \gamma_k \|\tilde{P}_k u\|^2_{H^2(D)} + \alpha \|\theta_k u\|^2_{H^2(D)} \tag{4.30}$$

where as in (3.18) $\gamma_k$ can be chosen such that $\gamma_k = \tilde{\gamma}/(1 + k^4)$ where $\tilde{\gamma}$ depends only on $n$ and $D$. From the expression of $P_k u$ one sees that there exists a constant $c$ that depends only of $\chi$ such that

$$\|\tilde{P}_k u\|^2_{H^2(D)} \geq \|u\|^2_{H^2(D)} - 2c\|u\|_{H^2(D)} \|\theta_k u\|_{H^2} + \|\chi \theta_k u\|^2_{H^2(D)} \tag{4.31}$$

$$(\tilde{A}_k u, u)_{H^2(D)} \geq \gamma_k \|u\|^2_{H^2(D)} - 2c\gamma_k \|u\|_{H^2(D)} \|\theta_k u\|_{H^2} + \alpha \|\theta_k u\|^2_{H^2(D)}. \tag{4.32}$$

Let $\alpha_0 = \tilde{\gamma} c^2$. Since $\gamma_k < \tilde{\gamma}$, we observe that $(\gamma_k c)^2 < \gamma_k \alpha$ for all $k$ and $\alpha \geq \alpha_0$ and therefore the operator $A_k$ is strictly coercive for $\alpha \geq \alpha_0$.

We now prove the second assertion. We observe that

$$(\tilde{A}_k u, u)_{H^2(D)} + (\tilde{B}_k u, u)_{H^2(D)} = \mathcal{A}(\tilde{P}_k u, \tilde{P}_k u) + \alpha \|\theta_k u\|^2_{H^2(D)} + \mathcal{B}_k(\tilde{P}_k u, \tilde{P}_k u). \tag{4.33}$$

Therefore $(\tilde{A}_k u, u)_{H^2(D)} + (\tilde{B}_k u, u)_{H^2(D)} = 0$ implies that $\theta_k u = 0$ and

$$\mathcal{A}(\tilde{P}_k u, \tilde{P}_k u) + \mathcal{B}_k(\tilde{P}_k u, \tilde{P}_k u) = 0 \tag{4.34}$$

According to the previous section, there exists $k_0 > 0$ such that for all $0 < k < k_0$ if (4.34) holds then $\tilde{P}_k u = 0$. We conclude that $u = P_k u - \chi \theta_k u = 0$. □

THEOREM 4.3. *The set of transmission eigenvalues is discrete.*

*Proof.* The previous lemma proves in particular that for $\alpha$ sufficiently large $\tilde{A}_k$ is coercive in a neighborhood of the positive real axis (since $\tilde{A}_k$ is continuous with respect to $k$) and therefore invertible. In this neighborhood $\tilde{A}_k^{-1}$ is analytic and hence the operator $I + \tilde{A}_k^{-1}\tilde{B}_k$ depends analytically on $k$ and is injective for $k$ sufficiently small. The analytic Fredholm theory now shows that this operator is injective for all values of $k$ in this neighborhood except for at most a discrete set of values. □

**4.2.2. Existence of transmission eigenvalues.** We shall prove in this section that there exist at least one transmission eigenvalue and therefore we assume here that $k$ is positive. To this end, we observe that $k$ is a transmission eigenvalue if and only if the operator

$$A_k + B_k : V(D, D_0, k) \to V(D, D_0, k) \tag{4.35}$$

has a nontrivial kernel, where $A_k$ is the positive definite self-adjoint operator associated with the coercive bilinear form $\mathcal{A}(\cdot\,,\,\cdot)$ and $B_k$ is the compact self-adjoint operator associated with the bilinear from $\mathcal{B}(\cdot\,,\,\cdot)$. Define the operator $A_k^{-1/2}$ by $A_k^{-1/2}=\int_0^\infty \lambda^{-1/2}dE_\lambda$ where $dE_\lambda$ is the spectral measure associated with the positive definite operator $A_k$. In particular, $A_k^{-1/2}$ is also bounded, positive definite and self-adjoint. Hence it is obvious that $k$ is a transmission eigenvalue if and only if the operator

$$I_k + A_k^{-1/2}B_kA_k^{-1/2} : V(D,D_0,k) \to V(D,D_0,k) \tag{4.36}$$

has a nontrivial kernel. Note that $A_k^{-1/2}B_kA_k^{-1/2}$ is a compact self-adjoint operator.

Similarly to above, in order to avoid dealing with function spaces depending on $k$ we shall introduce this time the orthogonal projection oprator $P_k$ from $H_0^2(D)$ onto $V(D,D_0,k)$ and the corresponding injection $R_k : V(D,D_0,k) \to H_0^2(D)$ and then consider the operator

$$I + R_kA_k^{-1/2}B_kA_k^{-1/2}P_k : H_0^2(D) \to H_0^2(D). \tag{4.37}$$

To show that a $k>0$ is a transmission eigenvalue it suffices to show that for this $k$ the kernel of the operator (4.37) is nontrivial since the injectivity of (4.36) implies the injectivity of (4.37). Indeed let $u \in H_0^2(D)$ be such that $(I+R_kA_k^{-1/2}B_kA_k^{-1/2}P_k)u = 0$. Then $u = P_ku + (u-P_ku)$ where $P_ku \in V(D,D_0,k)$ and $w := u - P_ku$ is in the orthogonal complement of $V(D,D_0,k)$. In particular $P_kw = 0$ and

$$\begin{aligned} 0 &= (u,\,w)_{H^2} + \left(R_kA_k^{-1/2}B_kA_k^{-1/2}P_ku,\,w\right)_{H^2} \\ &= (w,\,w)_{H^2} + \left(A_k^{-1/2}B_kA_k^{-1/2}P_ku,\,P_kw\right)_{H^2} = \|w\|^2_{H^2}, \end{aligned} \tag{4.38}$$

whence $w=0$. The injectivity of $A_k^{-1/2}B_kA_k^{-1/2}$ now implies that $P_ku=0$ which means that (4.37) is injective.

Our next goal is to prove that $R_kA_k^{-1/2}B_kA_k^{-1/2}P_k$ depends continuously on $k$. To this end, in the following two lemmas we prove the continuity of the projection operator $P_k$.

Lemma 4.4. *Assume that $0<k\le k_0$. Then there exists a constant $C(k_0)$ such that*

$$\|u-P_ku\|_{H^2(D)} \le C(k_0)\|\Delta u + k^2u\|_{L^2(D_0)} \tag{4.39}$$

*for all $u \in H_0^2(D)$.*

*Proof.* Let $u\in H_0^2(D)$ and let $\tilde{P}_k$ be the operator defined by (4.24). Then

$$\begin{aligned} \|P_ku-u\|_{H^2(D)} &\le \|\tilde{P}_ku-u\|_{H^2(D)} = \|\chi\theta_ku\|_{H^2(D)} \\ &\le C\|\theta_ku\|_{H^2(D)} \le CC(k)\|\Delta u + k^2u\|_{L^2(D_0)}. \end{aligned} \tag{4.40}$$

Since $\theta_k$ depends continuously on $k$, one can bound $CC(k)$ by a constant that only depends on $k_0$ for all $k\le k_0$ $\square$

Theorem 4.5. *The projection operator $P_k : H_0^2(D) \to V(D,D_0,k)$ is continuous with respect to $k>0$.*

*Proof.* Let $k$ and $k'$ be positive numbers less than $k_0$ and let $u$ be in $H_0^2(D)$. Set $u_k := P_k u$ and $u_{k'} := P_{k'} u$. Then

$$\|u_k - u_{k'}\|^2_{H^2(D)} = \|P_k(u_k - u_{k'})\|^2_{H^2(D)} + \|(I - P_k)(u_k - u_{k'})\|^2_{H^2(D)}. \tag{4.41}$$

On the one hand, using Lemma 4.4,

$$\begin{aligned} \|(I - P_k)(u_k - u_{k'})\|_{H^2(D)} &= \|(I - P_k)u_{k'})\|_{H^2(D)} \\ &\leq C(k_0)\|\Delta u_{k'} + k^2 u_{k'}\|_{L^2(D_0)} \\ &= C(k_0)|k^2 - k'^2|\|u_{k'}\|_{L^2(D_0)} \\ &\leq C(k_0)|k^2 - k'^2|\|u\|_{H^2(D)} \end{aligned} \tag{4.42}$$

and, on the other hand,

$$\begin{aligned} \|P_k(u_k - u_{k'})\|^2_{H^2(D)} &= (P_k(u_k - u_{k'}), P_k(u_k - u_{k'}))_{H^2(D)} \\ &= (P_k(u_k - u_{k'}), u_k - u_{k'})_{H^2(D)} \\ &= (P_k(u_k - u_{k'}), u_k - u + u - u_{k'})_{H^2(D)} \\ &= (P_k(u_k - u_{k'}), u - u_{k'})_{H^2(D)} \\ &= ((I - P_{k'})P_k(u_k - u_{k'}), u)_{H^2(D)}. \end{aligned} \tag{4.43}$$

Applying Lemma 4.4, we have

$$\begin{aligned} \|(I - P_{k'})P_k(u_k - u_{k'})\|_{H^2(D)} &\leq C(k_0)\|(\Delta + k'^2)P_k(u_k - u_{k'})\|_{L^2(D_0)} \\ &= C(k_0)|k'^2 - k^2|\|P_k(u_k - u_{k'})\|_{L^2(D_0)} \\ &\leq C(k_0)|k'^2 - k^2|\|u_k - u_{k'}\|_{H^2(D)}. \end{aligned} \tag{4.44}$$

Therefore we conclude that

$$\|P_k(u_k - u_{k'})\|^2_{H^2(D)} \leq C(k_0)|k'^2 - k^2|\|u_k - u_{k'}\|_{H^2(D)}\|u\|_{H^2(D)}. \tag{4.45}$$

Using the previous estimates in the first equality yields

$$\|u_k - u_{k'}\|_{H^2(D)} \leq \frac{\sqrt{5}+1}{2} C(k_0)|k'^2 - k^2|\|u\|_{H^2(D)} \tag{4.46}$$

which proves in particular that $k \to P_k u$ is continuous. ◻

Corollary 4.6. *The operator valued function* $k \to R_k A_k^{-1/2} B_k A_k^{-1/2} P_k \in \mathcal{L}(H_0^2(D), H_0^2(D))$ *is continuous with respect to* $k > 0$.

*Proof.* It is more convienient to introduce the operator $\hat{A}_k = (I - R_k P_k) + R_k A_k P_k : H_0^2(D) \to H_0^2(D)$ and observe that it is a selfadjoint and positive definite operator and also depends continuously on $k$. We deduce that the positive definite selfajoint operator $\hat{A}_k^{-1/2}$ also continuously depends on $k$ (this follows immediately from the continuity of $P_k$ and the definition of $A_k$). Simple calculations show that $\hat{A}_k^{-1/2} = (I - R_k P_k) + R_k A_k^{-1/2} P_k$ and therefore

$$R_k A_k^{-1/2} B_k A_k^{-1/2} P_k = \hat{A}_k^{-1/2} R_k B_k P_k \hat{A}_k^{-1/2}.$$

We now only need to prove that $k \to R_k B_k P_k$ is continuous. The latter follows immediately from the expression for $\mathcal{B}_k$ and the continuity of $P_k$. ◻

From the max-min principle for the eigenvalues $\lambda(k)$ of the compact and selfadjoint operator $R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ and from Corollary 4.6 we can conclude that $\lambda(k)$ is a continuous function of $k$. The proof of the existence of transmission eigenvalues is based on the following theorem which was first proved in [13] and we include here for reader's convenience.

Theorem 4.7. *Let* $T_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$. *Assume that*

*1) There is a $\kappa_0$ such that $I + T_{\kappa_0}$ is positive on $H^2_0(D)$.*
*2) There is a $\kappa_1 > \kappa_0$ such that $I + T_{\kappa_1}$ is non positive on a $p$-dimensional subspace $W_k$ of $H^2_0$.*

*Then there are $p$ transmission eigenvalues in $[\kappa_0, \kappa_1]$ counting their multiplicity.*

*Proof.* If $I + T_{\kappa_0}$ is positive it means that it has a trivial kernel and so does the operator given by (4.36), whence $\kappa_0$ is not a transmission eigenvalue. In particular 0 is not an eigenvalue of $I + T_{\kappa_0}$ and hence all eigenvalues satisfy $\lambda_j(\kappa_0) > 0$. Next the assumption *2)* guaranties that the operator $I + T_{\kappa_1}$ has $p$ negative eigenvalues $\lambda_j(\kappa_1) < 0$ for $j = 1 \dots p$, counting the multiplicity. The continuity of the spectrum implies that there is a $k \in [\kappa_0, \kappa_1]$ such that $\lambda_j(k) = 0$, for each $1 \le j \le p$. This means that $I + T_k$ has a non trivial kernel for those $p$ values of $k$, whence we can conclude that there are $p$ transmission eigenvalues counting the multiplicity. □

Let us denote by $\mu_p(D \setminus \overline{D}_0) > 0$ the $p - th$ clamped plate eigenvalue (counting the multiplicity) on $D/\overline{D}_0$ and set

$$\theta_p(D \setminus \overline{D}_0) := 4\frac{\mu_p(D \setminus \overline{D}_0)^{1/2}}{\lambda_1(D)} + 4\frac{\mu_p(D \setminus \overline{D}_0)}{\lambda_1(D)^2} \tag{4.47}$$

where again $\lambda_1(D)$ is the first eigenvalue of $-\Delta$ in $D$.

THEOREM 4.8. *Let $n \in L^\infty(D)$ satisfying either one of the following assumptions for $x \in D \setminus \overline{D}_0$*
*1) $1 + \theta_p(D \setminus \overline{D}_0) \le n_* \le n(x) \le n^* < \infty$,*
*2) $0 < n_* \le n(x) \le n^* < \frac{1}{1+\theta_p(D \setminus \overline{D}_0)}$.*

*Then, there exist $p$ transmission eigenvalues (counting the multiplicity).*

*Proof.* First we assume that the assumption 1) holds. Then from (4.7) and the fact that $A_k + B_k$ is positive if and only if $I + R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ is positive we have that $k > 0$ such that $k^2 > \lambda_1(D)/n^*$ is not a transmission eigenvalue and for those $k > 0$ the assumption *1)* of Theorem 4.7 is valid.

Next set $M = \sup_{D \setminus \overline{D}_0}(\frac{1}{n-1}) = \frac{1}{n_*-1}$. Then, restricting ourselves to functions in $V(D, D_0, k)$ such that $\|u\|_{L^2} = 1$, and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(A_k u + B_k u,\, u)_{H^2(D)} = &\int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}|\Delta u|^2\,dx + k^4 \int\limits_{D\setminus\overline{D}_0} \frac{n}{n-1}|u|^2\,dx + k^4\int\limits_{D_0}|u|^2\,dx \\ &+k^2 \int\limits_{D\setminus\overline{D}_0} \frac{1}{n-1}(\overline{u}\Delta u + u\Delta\overline{u})\,dx - k^2\int\limits_D |\nabla u|^2\,dx \\ &\le M\|\Delta u\|^2_{L^2(D\setminus\overline{D}_0)} + k^4(1+M) + 2k^2 M\|\Delta u\|_{L^2(D\setminus\overline{D}_0)} - k^2\|\nabla u\|^2_{L^2(D)}.\end{aligned} \tag{4.48}$$

Applying the Poincaré inequality to $u \in H^1_0(D)$ one has

$$\|\nabla u\|^2_{L^2(D)} \ge \lambda_1(D). \tag{4.49}$$

Now denote by $V_p$ the $p$ dimensional eigenspace associated with the first $p$ clamped plate eigenvalues in the doubly connected domain $D \setminus \overline{D}_0$. In particular, for any $u \in V_p$ such that $\|u\|_{L^2(D\setminus\overline{D}_0)} = 1$ then $\|\Delta u\|^2_{L^2(D\setminus\overline{D}_0)} \le \mu_p$. If $\tilde{u}$ is the extension by zero of $u \in V_p$ to the whole of $D$ then since the Cauchy data of $u$ are zero on $\partial D_0$ we have that $\tilde{u} \in V(D, D_0, k) \subset H^2_0(D)$. Hence for these $p$ linearly independent functions $\tilde{u}$ we have

$$\begin{aligned}(A_k\tilde{u} + B_k\tilde{u},\, \tilde{u})_{H^2(D)} \le\; &k^4(1+M) - k^2\left(\lambda_1(D) - 2M\mu_p(D\setminus\overline{D}_0)^{1/2}\right) \\ &+ M\mu_p(D\setminus\overline{D}_0)\end{aligned} \tag{4.50}$$

for any $k^2 > 0$. In particular, the value of $k_1^2 = \frac{\lambda_1(D) - 2M\mu_p(D\setminus\overline{D}_0)^{1/2}}{2+2M}$ minimizes the right hand side, whence we obtain

$$(A_k\tilde{u} + B_k\tilde{u},\, \tilde{u})_{H_0^2} \leq -\frac{\left(\lambda_1(D) - 2M\mu_p(D\setminus\overline{D}_0)^{1/2}\right)^2}{4+4M} + M\mu_p(D\setminus\overline{D}_0) \qquad (4.51)$$

which becomes non positive if $M \leq \frac{\lambda_1(D)^2}{4\mu_p(D\setminus\overline{D}_0)^{1/2}\left(\lambda_1(D)+\mu_p(D\setminus\overline{D}_0)^{1/2}\right)}$ which means that

$$\inf_{D\setminus\overline{D}_0}(n) \geq 1 + 4\frac{\mu_p(D\setminus\overline{D}_0)^{1/2}}{\lambda_1(D)} + 4\frac{\mu_p(D\setminus\overline{D}_0)}{\lambda_1(D)^2} = 1 + \theta_p(D\setminus\overline{D}_0). \qquad (4.52)$$

Since $A_k + B_k$ and $I + R_kA_k^{-1/2}B_kA_k^{-1/2}P_k$ coincide in $V(D, D_0, K)$ we conclude that the assumption *2)* of Theorem 4.7 is valid and therefore the result follows from Theorem 4.7.

Now we assume that the assumption 2) holds. Then from Section 4.1 and the fact that $A_k + B_k$ is positive if and only if $I + R_kA_k^{-1/2}B_kA_k^{-1/2}P_k$ is positive we have that $k > 0$ such that $k^2 > \lambda_1(D)$ are not transmission eigenvalues and for those $k > 0$ the assumption *1)* of Theorem 4.7 is valid.

Let us now define $M = \max\left\{sup_{D\setminus\overline{D}_0}\frac{n}{1-n}, 1\right\} = \max\left\{\frac{n^*}{1-n^*}, 1\right\}$ and observe that $\frac{1}{1-n} \leq M+1$. Then doing the same type of calculations as above assuming that $u \in V_p$ and $\|u\|_{L^2}^2 = 1$, and denoting by $\tilde{u} \in V(D, D_0, k)$ the extension by zero to the whole of $D$, we obtain

$$\begin{aligned}\left(\tilde{A}_k\tilde{u} + B_k\tilde{u},\, \tilde{u}\right)_{H^2} &= \int\limits_{D\setminus\overline{D}_0}\frac{1}{1-n}|\Delta\tilde{u}|^2\,dx + k^4\int\limits_{D\setminus\overline{D}_0}\frac{n}{1-n}|\tilde{u}|^2\,dx + k^4\int\limits_{D_0}|\tilde{u}|^2\,dx\\ &\quad + k^2\int\limits_{D\setminus\overline{D}_0}\frac{n}{1-n}(\overline{\tilde{u}}\Delta\tilde{u} + \tilde{u}\Delta\overline{\tilde{u}})\,dx - k^2\int\limits_{D}|\nabla\tilde{u}|^2\,dx \qquad (4.53)\\ &\leq (M+1)\|\Delta u\|_{L^2}^2 + k^4M + 2k^2M\|\Delta u\|_{L^2} - k^2\|\nabla u\|_{L^2}^2\\ &\leq k^4M - k^2\left(\lambda_1(D) - 2M\mu_p(D\setminus\overline{D}_0)^{1/2}\right) + (M+1)\mu_p(D\setminus\overline{D}_0).\end{aligned}$$

The minimizing value of $k^2$ of the right hand side is now $k_1^2 = \frac{\lambda_1(D)-2M\mu_p(D\setminus\overline{D}_0)^{1/2}}{2M}$ which gives

$$(A_k\tilde{u} + B_k\tilde{u},\, \tilde{u})_{H_0^2} \leq -\frac{\left(\lambda_1(D) - 2M\mu_p(D\setminus\overline{D}_0)^{1/2}\right)^2}{4M} + (M+1)\mu_p(D\setminus\overline{D}_0). \qquad (4.54)$$

Hence the latter becomes non positive if $M \leq \frac{\lambda_1(D)^2}{4\mu_p(D\setminus\overline{D}_0)^{1/2}\left(\lambda_1(D)+\mu_p(D\setminus\overline{D}_0)^{1/2}\right)}$ which means that $\sup_{D\setminus\overline{D}_0}(n) \leq 1/(1+\theta_p(D\setminus\overline{D}_0))$. Consequently if assumption 2) holds then $A_k + B_k$ is non positive on a $p$ dimensional subspace of $V(D, D_0, k)$ and so does $I + R_kA_k^{-1/2}B_kA_k^{-1/2}P_k$ on on a $p$ dimensional subspace of $H_0^2(D)$ and the result is proven in this case again by an application of Theorem 4.7
$\square$

Remark 4. From the proof of the above estimate we have that the first transmission eigenvalue satisfies

$$\frac{\lambda_1(D)}{\sup_{D\setminus\overline{D}_0}(n)} \leq k_1^2 < \frac{\lambda_1(D) - 2M\mu_1(D\setminus\overline{D}_0)^{1/2}}{2+2M} \qquad (4.55)$$

where $M = 1/(\inf_{D\setminus\overline{D}_0} n - 1)$, provided that

$$\inf_{D\setminus\overline{D}_0}(n) \geq 1 + 4\frac{\mu_1(D\setminus\overline{D}_0)^{1/2}}{\lambda_1(D)} + 4\frac{\mu_1(D\setminus\overline{D}_0)}{\lambda_1(D)^2} \tag{4.56}$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue for the Laplasian on $D$ and $\mu_1(D\setminus\overline{D}_0)$ is the first clamped plate eigenvalue on $D/\overline{D}_0$.

**Acknowledgments.** This research was initiated while F.C. and D.C. were visiting Ecole Polytechnique supported by the associate team ISIP of INRIA-UDEL. The hospitality of this institution is gratefully acknowledged. The research of F. C. and D.C. was supported in part by the U.S. Air Force Office of Scientific Research under Grant FA9550-08-1-0138. We also thank the referees for their careful reading of our paper which has resulted in a much improved paper.

REFERENCES

[1] F. Cakoni, M. Cayören and D. Colton, *Transmission eigenvalues and the nondestructive testing of dielectrics*, Inverse Problems, 26 (2008) paper 065016.

[2] F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Springer, Berlin, 2006.

[3] F. Cakoni and H. Haddar, *On the existence of transmission eigenvalues in an inhomogenous medium*, Applicable Analysis (to appear).

[4] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 2nd edition, Vol. 93 in Applied Mathematical Sciences, Springer, New York, 1998.

[5] D. Colton and R. Kress *Integral equation methods in scattering theory*, John Wiley & Sons Inc., New York, 1983.

[6] D. Colton and P. Monk, *The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium*, Quart. Jour. Mech. Applied Math. 41 (1988), pp. 97-125.

[7] D. Colton, L. Päivärinta and J. Sylvester, *The interior transmission problem*, Inverse Problems and Imaging, 1 (2007) pp. 13-28.

[8] A. Henrot, *Extremum Problems for Eigenvalues of Elliptic Operators*, Birkhause Verlag, Basel, 2006.

[9] A. Kirsch *The denseness of the far field patterns for the transmission problem*, IMA Jour. Appl. Math. 37 (1986), pp. 213-225.

[10] A. Kirsch and N. Grinberg, *The Factorization Method for Inverse Problems*, Oxford University Press, Oxford, 2008

[11] J. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications* Springer-Verlag, New York, 1972.

[12] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.

[13] L. Päivärinta and J. Sylvester, *Transmission eigenvalues*, SIAM J. Math. Anal., 40 (2008) pp. 738-753.

[14] B. P. Rynne and B.D. Sleeman, *The interior transmission problem and inverse scattering from inhomogeneous media*, SIAM J. Math. Anal., 22 (1992) pp. 1755-1762.

[15] N. Weck *Approximation by Herglotz wave functions*, Math. Methods Appl. Sci., 27 (2004), 155–162.