Groups with One Nonlinear Irreducible Brauer Character are Solvable

Xiaoyou Chen

College of Science
Henan University of Technology
Zhengzhou 450001, China
cxymathematics@hotmail.com

Jiwen Zeng

School of Mathematical Sciences
Xiamen University, Xiamen 361005, China
jwzeng@xmu.edu.cn

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Abstract

Let G be a finite group and p be a prime divisor of $|G|$. Denote by $\text{IBr}(G)$ and $\text{LBr}(G)$ the sets of irreducible p -Brauer characters and linear p -Brauer characters of G , respectively. If all the irreducible p -Brauer characters of G are linear, that is, $\text{IBr}(G) = \text{LBr}(G)$, then G is solvable. When G has only one nonlinear irreducible p -Brauer character, it is proved in this note that G is solvable. If G has two nonlinear irreducible p -Brauer characters, however, G is not necessarily solvable.

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1 Introduction

Let G be a finite group and G' be the derived subgroup of G . Denote by $\text{Irr}(G)$ and $\text{Lin}(G)$ the sets of irreducible (complex) characters and linear (complex)

characters of G , respectively. If all the irreducible characters of G are linear, that is, $\text{Irr}(G) = \text{Lin}(G)$, then G is abelian. G. Seitz and J. Pelikán proved that finite groups with only one nonlinear irreducible character are solvable in [7] and [6], respectively. They classified finite groups with only one nonlinear irreducible character from different views. G. Seitz proved that a finite group G with only one nonlinear irreducible character if and only if

- (i) $|G| = 2^k$, k is odd and $G' = \mathbf{Z}(G)$ with order 2, or
- (ii) G is the group of all transformations $v \rightarrow av + b$ on a finite field $GF(q)$ of order $q > 2$, where $a, b \in GF(q)$ and $a \neq 0$.

And J. Pelikán gave a full description of such groups of those two cases in detail. Also, K. Motose [4] considered finite groups with only one nonlinear irreducible character. J. Pelikán and K. Motose both proved that those groups are isomorphic to affine groups over finite fields or central products of dihedral groups D_8 and quaternion groups Q_8 .

Let G be a finite group and p be a prime divisor of $|G|$. Let G^0 be the set of p -regular elements, that is, $G^0 = \{g \in G \mid p \nmid o(g)\}$. Denote by $\text{cl}(G)$ and $\text{cl}(G^0)$ the sets of conjugacy classes and p -regular classes of G , respectively. And we write $\text{IBr}(G)$ and $\text{LBr}(G)$ for the sets of irreducible p -Brauer characters and linear p -Brauer characters of G , respectively. For notational convenience, once a prime number p is chosen we simply write Brauer characters for p -Brauer characters and regular classes for p -regular classes. For other notations and terminologies, one can refer to [3] and [5].

Inspired by [7] and [6], we consider in this note the case when a finite group G is with the number of nonlinear irreducible Brauer characters not more than 2. In fact, if all the irreducible Brauer characters of G are linear, then G is solvable. Also, we have the following.

Theorem 1.1. *Let G be a finite group and p be a prime divisor of $|G|$. If G has exactly one nonlinear irreducible Brauer character, then G is solvable.*

If $|\text{IBr}(G)| = |\text{LBr}(G)| + 2$, however, we cannot have that G is necessarily solvable. For instance, let A_5 be the alternating group on five letters and $p = 5$; then A_5 has two nonlinear irreducible Brauer characters. And that is the only one finite simple group which we could find up to date to have the property that it has only two nonlinear irreducible Brauer characters.

2 Proofs

It is known by character theory that the number of all linear characters of a finite group G is the index of the derived subgroup G' in G , that is, $|G : G'| = |\text{Lin}(G)|$. Similarly, for p -Brauer characters, we have the following.

Lemma 2.1. *Let G be a finite group and H be the smallest normal subgroup such that G/H is an abelian p' -group, that is, $H = G'\mathbf{O}^{p'}(G)$, where p is a prime divisor of $|G|$. Then*

- (a) ${}^0 : \text{Irr}(G/H) \rightarrow \text{LBr}(G)$, $\chi \mapsto \chi^0$ is a bijection;
- (b) $|\text{LBr}(G)| = |G : G'|_{p'}$, where $|G : G'|_{p'}$ denotes the p' -part of $|G : G'|$;
- (c) $\text{IBr}(G) = \text{LBr}(G)$ if and only if $P = H$, where P is the Sylow p -subgroup of G ;
- (d) G is solvable if $\text{IBr}(G) = \text{LBr}(G)$.

Proof. Let $\lambda \in \text{LBr}(G)$ be afforded by the F -representation $\Lambda : G \rightarrow F^\times$, where F is the algebraic closure of the prime field $\mathbb{Z}_p$ and $F^\times = F - \{0\}$. Then $G/\ker \lambda = G/\ker \Lambda \lesssim F^\times$. By Lemma 2.1 of [5], it is known that the field F does not contain nontrivial p -elements. It follows that $G/\ker \lambda$ is a cyclic p' -group and then $H \subseteq \ker \lambda$. Thus $\lambda \in \text{IBr}(G/H)$. Since G/H is an abelian p' -group, we have that $\text{Lin}(G/H) = \text{Irr}(G/H) = \text{IBr}(G/H) \subseteq \text{LBr}(G)$ and hence $\text{IBr}(G/H) = \text{LBr}(G)$. Note that $\chi_{G^0} = \chi^0 \in \text{LBr}(G)$ for every $\chi \in \text{Irr}(G/H)$. And then ${}^0 : \text{Irr}(G/H) \rightarrow \text{LBr}(G)$ is a bijection. In fact, since $\text{Irr}(G/H)$ and $\text{LBr}(G)$ are groups, it follows that 0 is a group isomorphism. Notice that G/G' is nilpotent. It follows that

$$|G : G'|_{p'} = |G/G' / H/G'| = |G/H| = |\text{LBr}(G)|$$

and the proofs of (a) and (b) are complete.

If $\text{IBr}(G) = \text{LBr}(G)$, then $G/\ker \varphi$ is a cyclic p' -group for each $\varphi \in \text{IBr}(G)$. Thus $G'\mathbf{O}^{p'}(G) \leq \ker \varphi$ and then $G'\mathbf{O}^{p'}(G) \leq \bigcap_{\varphi \in \text{IBr}(G)} \ker \varphi = \mathbf{O}_p(G)$ by Lemma 2.1 of [1]. Since

$$P \leq G'\mathbf{O}^{p'}(G) \leq \mathbf{O}_p(G) \leq P,$$

it follows that $P = H$. Conversely, if $P = H$, then for every $\varphi \in \text{IBr}(G)$ we conclude that $\varphi \in \text{IBr}(G/P) = \text{IBr}(G/H)$. Note that G/P is an abelian p' -group. Thus $\text{IBr}(G) = \text{LBr}(G)$ by (a) and the proof of (c) is complete.

If $\text{IBr}(G) = \text{LBr}(G)$, it follows that $H = G'\mathbf{O}^{p'}(G)$ is the Sylow p -subgroup of G . Also, G/H is an abelian p' -group and then G is solvable. The proof is complete. $\square$

If all the irreducible Brauer characters of the finite group G are linear, we have that G has a Sylow tower by Lemma 2.1. We know that a finite supersolvable group always has a Sylow tower. Is G supersolvable if $\text{IBr}(G) = \text{LBr}(G)$? The answer is negative. For example, let A_4 be the alternating group on four letters and $p = 2$; then all the irreducible Brauer characters of A_4 are linear, but A_4 is not supersolvable.

Now, we list the following theorem which contains Theorem 1.1 and give the proof.

Theorem 2.2. *Let G be a finite group and p be a prime divisor of $|G|$. Suppose that G have only one nonlinear irreducible Brauer character. Then we have the following.*

- (i) *If $1 < K \triangleleft G$ and K is not a p -group, then $G'\mathbf{O}^{p'}(G) \subseteq PK$, where $P \in \text{Syl}_p(G)$.*
- (ii) *G is solvable.*

Proof. (i) Let φ be the nonlinear irreducible Brauer character of G . Notice that G can act by conjugation on $\text{IBr}(K)$, that is, if $\theta \in \text{IBr}(K)$, $g \in G$, then $\theta^g(h) = \theta(h^{g^{-1}})$. Since K is not a p -group, it follows that G has at least two orbits on $\text{IBr}(K)$. Now, by Clifford's theorem, we can take $\theta \in \text{IBr}(K) - \text{IBr}(\varphi_K)$, where $\text{IBr}(\varphi_K)$ denotes the set of all irreducible constituents of φ_K . Again by Clifford's theorem, we conclude that φ is not a constituent of θ^G , and then all of the irreducible constituents of θ^G are linear. Therefore $\text{IBr}(\theta^G) \subseteq \text{LBr}(G) = \text{IBr}(G/G'\mathbf{O}^{p'}(G))$.

Let λ be a linear Brauer constituent of θ^G . Then $\lambda_K = \theta$. By Gallagher's theorem (see also Corollary 8.20 in [5]), it follows that $\text{IBr}(\theta^G) = \{\lambda\mu \mid \mu \in \text{IBr}(G/K)\}$ and therefore all of the irreducible Brauer characters of G/K are linear. Thus the Sylow p -subgroup PK/K of G/K is normal in G/K , where $P \in \text{Syl}_p(G)$. By Lemma 2.1(c), we have G/PK is an abelian p' -group, and then $G'\mathbf{O}^{p'}(G) \subseteq PK$.

(ii) Write $H = G'\mathbf{O}^{p'}(G)$. We first argue that each nonidentity element of G/H is a disjoint union of regular classes of G . Let $a, b \in G^0$ be conjugate. Then there exists an element $g \in G$ such that $a^g = b$ and thus $a^{-1}g^{-1}ag = a^{-1}b \in G' \subseteq H$. It follows that $a, b \in aH$, as wanted.

Now, since $|\text{IBr}(G)| = |\text{LBr}(G)| + 1 = |G : H| + 1$ by the proof of Lemma 2.1 and $|\text{IBr}(G)| = |\text{cl}(G^0)|$, it follows that H^0 consists of two regular classes of G which are $\{1\}$ and $H^0 - \{1\}$. Then the nonidentity regular elements of H have the same prime order. Therefore, $|H|$ is divisible by two distinct primes and by Burnside's p^aq^b theorem it follows that H is solvable. Thus G is solvable and the proof is complete. $\square$

When $\text{IBr}(G) = \text{LBr}(G)$, we have another explanation of H being the Sylow p -subgroup of G . Note that $|\text{IBr}(G)| = |G : H|$ and similar to the above proof it follows that H contains only one regular class $\{1\}$ of G . Thus H is a p -group and then H is the Sylow p -subgroup of G .

Note We do not require the condition that $\mathbf{O}_p(G) = 1$ to prove the solvability of G . Notice that Dolfi and Navarro [2] prove that the groups with one nonlinear irreducible Brauer character are solvable under the condition that $\mathbf{O}_p(G) = 1$. In general, if G has only one nonlinear irreducible Brauer character, we do not necessarily have that $\mathbf{O}_p(G) = 1$. For example, let $G = C_3 \rtimes C_4$ with order 12 and $p = 2$; then $|\text{IBr}(G)| = |\text{LBr}(G)| + 1$, but $\mathbf{O}_2(G) \cong C_2$.

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