

## Lorenz Bifurcation: Instabilities in Quasireversible Systems

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We describe the two generic instabilities which arise in quasireversible systems and show that their normal forms are the well-known real Lorenz equations and the Maxwell-Bloch equations. We present for the first time analytic predictions for the appearance of Lorenz chaos and we describe a simple mechanical system which experimentally displays this chaotic behavior.

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The study of instabilities plays a central role in the modern theory of dynamical systems [1] and allows one to describe in a universal way phenomena which belong to different fields [2]. In one parameter families of dissipative dynamical systems (i.e., in codimension one) only two local bifurcations occur generically: the saddle node and the Hopf bifurcations. The presence of symmetries changes this picture: for example, the reflection symmetry  $Z_2$  leads to the pitchfork bifurcation. In this Letter, we shall consider some consequences of time reversal symmetry [3] when it is weakly broken, a situation which occurs frequently in the macroscopic description of systems. In their usual versions, the fundamental laws of physics are time reversible but this symmetry disappears in the macroscopic description due to dissipative phenomena which can vary from weak perturbations (quasireversibility) to strong dominant effects. Some well-known examples of quasireversible behavior in mechanics, fluid mechanics, and optics are, respectively, the motion of planets in celestial mechanics, surface waves in water, and the laser. Generic or codimension-one instabilities in strictly reversible systems are associated with resonances which can occur at either zero or finite frequency. The reduced linear operator is then given by a semisimple Jordan block  $\alpha^2$  (we use the notation introduced by Arnold [4]), where  $\alpha = 0$  corresponds to zero frequency and  $\alpha = i\omega$  to finite frequency. The finite frequency resonance called “confusion of frequencies” by Rocard [5] occurs, for example, in lasers (see, for example, [6] and references therein).

The dissipation and the forcing break the reversibility symmetry and often their effect is to select particular solutions of the reversible system. However, in the situation studied here, the addition of nonreversible terms will lead to new behaviors. An important question is how the forcing occurs in “quasireversible” systems, i.e., reversible systems to which one adds small terms which violate time reversal invariance. We remark that for a large class of physical systems pure static forcing is involved, i.e., forcing which occurs at zero frequency. An important example is the laser where the injection of energy occurs through a population inversion. The previous observation is related to the fact that zero

frequency neutral modes provide a natural way to couple a reversible system with a static source of energy.

Our objective here is to study the weakly dissipative-forced unfolding of the reversible bifurcations  $(0^2)(0)$  and  $(i\omega)^2(0)$  whose Jordan blocks are

$$\begin{aligned} \mathcal{L}_{(0)^2(0)} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \mathcal{L}_{(i\omega)^2(0)} &= \begin{pmatrix} i\omega & 1 & 0 \\ 0 & i\omega & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (1)$$

We shall see that in these quasireversible systems the  $(0^2)(0)$  bifurcation in the presence of an additional  $Z_2$  symmetry is described by the well-known real Lorenz equations which are its normal form and that the  $(i\omega)^2(0)$  bifurcation has as its normal form the popular set of Maxwell-Bloch equations. The latter have been derived from microscopic physics by many authors to describe lasers [6]. We shall present a very simple mechanical system (a rotating pendulum) which is an example of the  $(0^2)(0)$  quasireversible bifurcation with reflection symmetry and displays Lorenz chaos. It should be noted that in reversible systems all modes are neutral, but generically they do not couple except a zero frequency mode which has a strong resonant coupling with the frequency modes. The study presented here is done in a region of the parameter space in the neighborhood of the instability point and in this domain we can make analytical predictions. However, this region where we find the usual codimension-one Lorenz-type chaos is the germ of a bigger domain in the space of parameters where Lorenz-type chaos persists and to which it can be generically extended. In this bigger region the system ceases to be quasireversible and we cannot make analytic calculations anymore since we are far away from the reversible instability. This is illustrated, for example, by the historical values of the parameters of the Lorenz equations [7] and also by the observed experimental behavior of the pendulum about which a preliminary report was presented in [8].

The  $(0)^2(0)$  instability is described by three real variables  $(x, y, z)$ . In the case in which we have reflection symmetry  $x \rightarrow -x, y \rightarrow -y, z \rightarrow z$  and if the time reversal transformation is  $t \rightarrow -t, x \rightarrow x, y \rightarrow -y, z \rightarrow z$ , the asymptotic supercritical normal form of this instability is given by

$$\dot{x} = y, \quad \dot{y} = \epsilon x - x^3 - xz, \quad \dot{z} = 0, \quad (2)$$

where  $\epsilon$  is the bifurcation parameter,  $\partial_t \sim O(\sqrt{\epsilon}), x \sim O(\sqrt{\epsilon}), y \sim O(\epsilon), z \sim O(\epsilon)$ . If we include dissipation in the unfolding, Eq. (2) becomes

$$\begin{aligned} \ddot{x} &= \epsilon x - x^3 - xz - \nu \dot{x}, \\ \dot{z} &= -\mu z + \eta x^2, \end{aligned} \quad (3)$$

where the unfolding parameters  $(\nu, \mu, \eta)$  are all  $O(\sqrt{\epsilon})$ . This set of differential equations is equivalent to a set of equations which has exactly the Lorenz form [7,9]. The change of variables

$$\begin{aligned} x &= \frac{\epsilon}{\sqrt{\sigma}} x', & \dot{x} &= \frac{\epsilon^2}{\sqrt{2}} (y' - x'), \\ z &= \epsilon^2 \left( z' - \frac{x'^2}{2\sigma} \right), \end{aligned} \quad (4)$$

and the rescalings  $t \equiv (\epsilon/\sqrt{\sigma})t', \nu = \epsilon \frac{\sigma+1}{\sigma}, \mu = \epsilon(b/\sqrt{\sigma}), \eta = [\epsilon(2\sigma - b)/\sqrt{\sigma}]$ , and  $\epsilon = \sqrt{R - 1}$ , transform Eqs. (3) in

$$\begin{aligned} \partial_t x &= \sigma(y - x), \\ \partial_t y &= Rx - y - xz, & \partial_t z &= -bz + xy. \end{aligned} \quad (5)$$

These equations are in the Lorenz form and the three new parameters are now of order 1. It should be noted, however, that the parameter region of the quasireversible limit which we consider here is far away from the ‘‘historical’’ region of parameters of the Lorenz model [9]. The principal advantage of the quasireversible limit of these equations is that it allows us to make analytic calculations. One knows that the reversible systems (2) has a planar homoclinic solution  $x_h = \sqrt{\epsilon} \sec h(t\sqrt{\epsilon}/2), z = 0$  (see Fig. 1). In the weakly dissipative unfolding, this homoclinic solution becomes weakly three dimensional. The Melnikov condition for the persistence of the homoclinic solution is

$$\nu = 2\eta \left( 1 - \frac{\mu}{\sqrt{\epsilon}} \frac{\int_{-\infty}^{\infty} dt x_h^2(t) \int_{-\infty}^t ds e^{\mu(s-t)} x_h^2(s)}{\int_{-\infty}^{\infty} dt (\partial_t x_h)^2} \right), \quad (6)$$

which reduces in lowest order in  $\mu/\sqrt{\epsilon}$  to  $\nu = 2\eta + O(\mu/\sqrt{\epsilon})$ . Since  $\nu$  is a dissipation it is necessarily positive and, consequently, the Melnikov condition can be satisfied only for  $\eta > 0$ . This result shows that chaotic behaviors of the Lorenz type are present in the asymptotic unfolding of the  $(0)^2(0)$  instability as it can be easily checked by numerical simulations of Eqs. (3). It is remarkable that a perturbation calculation can predict chaos, and the reason for this here is that due to the

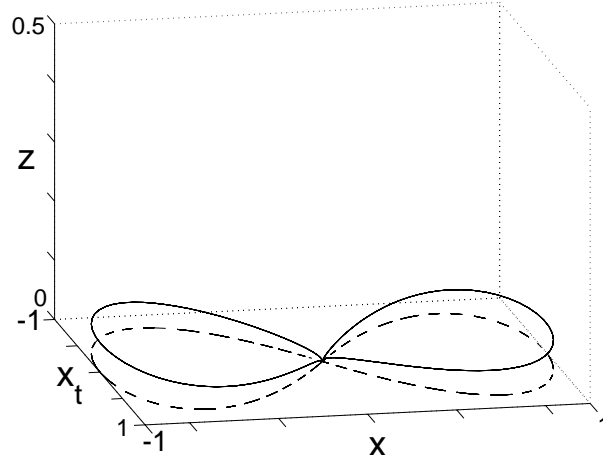


FIG. 1. Deformation of the planar homoclinic represented by a dashed line in the  $(x, x_t)$  plane. The perturbed homoclinic (solid line) becomes slightly three dimensional at the origin. In other places the original orbit is simply displaced upwards by an approximately constant value.

reflection symmetry in order to have chaos one needs only the original planar homoclinic to become slightly three dimensional (see Fig. 1), a situation which can be attained perturbatively. We note that the original planar homoclinic  $(x_h, z = 0)$  has equal eigenvalues  $\pm\sqrt{\epsilon}$  and in the perturbed situation the ratio of the absolute values of these eigenvalues (the smaller over the bigger one) will be slightly smaller than one and a small eigenvalue transversal to the  $(x, y)$  plane appears: then the usual analysis with a suitable Poincaré section [1] will give a mapping which shows that the strange attractor appears immediately with a large basin of attraction which explains why it is numerically easy to observe chaos as soon as one is near the relation  $\nu = 2\eta$ , i.e., when the unstable homoclinic disappears.

A nice consequence of these results is the experimental realization of the  $(0)^2(0)$  instability by a rotating pendulum which consists of a support which can rotate around a vertical axis with angular velocity  $\dot{\varphi}$  and a pendulum formed by a mass  $m$  at a distance  $l$  from the support which oscillates in a vertical plane fixed with respect to the support (Fig. 2). The system will dissipate by friction in the contacts and by the motion of the mass  $m$  in a fluid (say the air) through Stokes law. We inject energy through a constant torque  $\tau$  applied to the support. In dimensionless variables, the equations of this system are  $(I = l^2/ml^2)$ , where  $I'$  is the inertial moment of the support)

$$\ddot{\theta} = \frac{\sin(2\theta)}{2} \dot{\varphi}^2 - \sin(\theta) - \nu \dot{\theta},$$

$$\begin{aligned} [I + \sin^2(\theta)] \ddot{\varphi} &= -2 \sin(\theta) \cos(\theta) \dot{\varphi} \dot{\theta} \\ &\quad - \tilde{\mu}(\dot{\varphi} - \Omega) - \nu \dot{\varphi} \sin^2(\theta), \end{aligned} \quad (7)$$

where  $\tilde{\mu}$  and  $\nu$  are the damping coefficients ( $\nu$  is related to Stokes law), and we have written  $\tau = \tilde{\mu}\Omega$ . The system is described by four variables  $(\theta, \dot{\theta}, \dot{\varphi}, \varphi)$ , but  $\varphi$

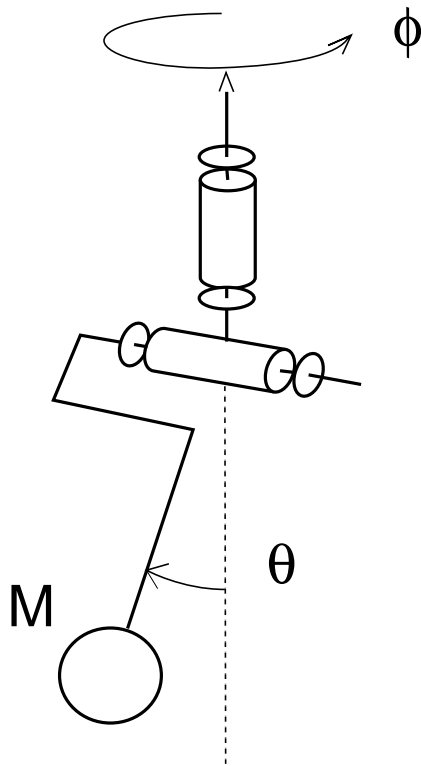


FIG. 2. Schematic representation of the pendulum.

is absent in (7) and the system is then a three dimensional one. The quantity  $P = [I + \sin^2(\theta)]\dot{\phi}$  is the total angular momentum with respect to the vertical axis which is conserved in the absence of dissipation and forcing, i.e., if  $\tilde{\mu} = \nu = 0$  (notice that since  $\tau = \tilde{\mu}\Omega$  the applied torque vanishes with the dissipation) in which case the system is reversible with respect to the time reversal transformation  $t \rightarrow -t, \theta \rightarrow \theta, \dot{\theta} \rightarrow -\dot{\theta}, \dot{\phi} \rightarrow \dot{\phi}$ . It is also integrable since  $P$  is constant and (7) reduces to

$$\ddot{\theta} = \frac{\sin(2\theta)}{2[I + \sin^2(\theta)]^2} P^2 - \sin(\theta). \quad (8)$$

In Eq. (7) for  $\nu = 0$ , we see that for  $\Omega < 1$  the stationary solution  $(\theta = \dot{\theta} = 0, \dot{\phi} = \Omega)$  is stable and it becomes marginal for  $\Omega = 1$ . In the quasireversible case, the critical spectrum has tree eigenvalues with a Jordan block of dimension 2, i.e., we have the  $(0)^2(0)$  instability. The asymptotic normal form of (7) then has the form (3) and it can be explicitly written as

$$\begin{aligned} \dot{\theta}' &= \epsilon\theta' - \nu\dot{\theta}' - \theta'\zeta' - \theta'^3, \\ \dot{\zeta}' &= -\mu\zeta' + \eta\theta'^2, \end{aligned} \quad (9)$$

where  $\epsilon = \Omega^2 - 1$ ,  $\mu = \tilde{\mu}/I$ ,  $\eta \equiv [12\Omega^2(\nu - \mu)]/[4(\Omega^2 - 1)I + 12\Omega^2]$ , and  $\dot{\phi} = \Omega - \zeta'/2\Omega - \theta'^2\{6I\Omega/[I(4\Omega^2 - 1) + 2\Omega^2]\}$ ,  $\theta = (\theta'\sqrt{6I})/\sqrt{4\Omega^2 - 1 + 12\Omega^2}$ .

Melnikov condition (6) tells us that we need  $\nu > \mu$  in order to have chaos, and we see then that the Stokes dissipation of the pendulum is essential for the chaotic behavior. Close to the instability ( $\Omega \approx 1$ ) the Melnikov

condition for the persistence of the homoclinic solution is  $\nu = \frac{6\mu}{4-\nu}$  and preliminary experimental observations exhibit the predicted chaotic behavior [8]. We have done numerical simulation of the original Eqs. (7) in the chaotic regime predicted by our analytic calculation and we have found the well-known scenario described by Sparow [7]. In Fig. 3 we show a simulation in which the characteristic aspect of the Lorenz attractor can be recognized.

When one has no reflection symmetry, Eqs. (3) are replaced by (we keep the same time reversal symmetry)

$$\begin{aligned} \dot{x} &= \epsilon \pm x^2 + azx \pm z^2 - \nu\dot{x}, \\ \dot{z} &= \delta - \mu z + \eta x, \end{aligned} \quad (10)$$

where  $\epsilon$  is the bifurcation parameter and  $(\delta, \eta, \nu, \mu)$  are related to dissipative and energy injection terms. In the reversible limit, one has again a plane homoclinic orbit and generic arguments indicate that Shilnikov-type chaos should be observed in Eqs. (10). We have checked this by numerical simulation. Since we still have a planar homoclinic, we can write the Melnikov condition for its persistence in the vicinity of the reversible limit. However, this information is not useful now since the Shilnikov scenario needs incoming complex eigenvalues and is then far away from our perturbed system.

In the case of the  $(i\omega)^2(0)$  instability the critical variables parametrizing the center manifold can be chosen

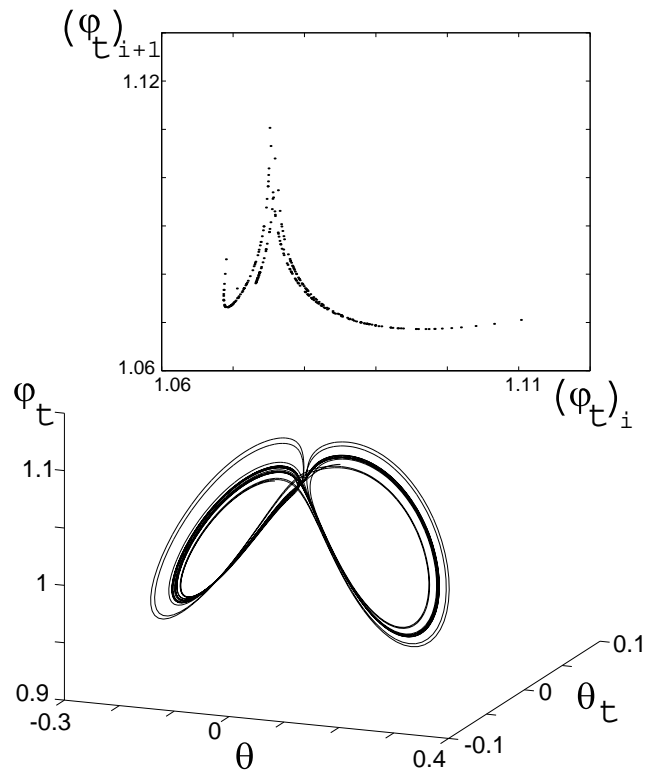


FIG. 3. The characteristic Lorenz map (top) and the phase space trajectory (bottom) obtained from Eqs. (7) for the parameter values  $\Omega = 1.4142$ ,  $\nu = 0.1210$ ,  $I = 0.3770$ , and  $\mu = 0.0037$ .

as two complex amplitudes ( $A, B$ ) and a real amplitude  $z$ . The global characterization of normal forms in [10] gives

$$\begin{aligned} A_t &= i\Omega A + B, & z_t &= f(|A|^2, i(\overline{A}B - \overline{A}B), z, \{\lambda\}), \\ B_t &= i\Omega B + Ag(|A|^2, \overline{A}B - \overline{A}B, z, \{\lambda\}) \\ &+ Bh(|A|^2, \overline{A}B - \overline{A}B, z, \{\lambda\}), \end{aligned} \quad (11)$$

where  $f$  is a real polynomial function, and  $g$  and  $h$  are complex polynomials of their arguments. If we impose invariance under the time reversal transformation  $t \rightarrow -t$ ,  $A \rightarrow \overline{A}$ ,  $B \rightarrow -\overline{B}$ ,  $z \rightarrow z$ , we conclude that in Eqs. (11) the function  $g$  is real,  $h$  is pure imaginary, and  $f$  vanishes thus reducing the third equation to  $z_t = 0$ . We see then that the neutral mode appears as a conserved quantity to any order of the normal form, i.e.,  $z$  is constant, and if we consider  $|z| \ll 1$ , the asymptotic reversible normal form reduces to

$$A_{tt} + ZA \pm |A|^2 A = 0, \quad Z_t = 0, \quad (12)$$

where we have put  $A \equiv A'e^{i\Omega t}$ ,  $B \equiv A'_t e^{i\Omega t}$  (rotating reference frame) and omitted the primes. The instability in (12) occurs when one chooses positive values of the conserved quantity  $z$ . In the case of a laser, in the unrealistic reversible limit, the population inversion is such a conserved quantity. The presence of this additional mode thus preserves the integrability of the normal form [11]. As in the  $(0)^2(0)$  case, this important property fails as soon as the forcing and the dissipation are taken into account since the dynamical system has then a higher dimension. Adding the small dissipative and forcing terms consistent with the general form (11) and doing some scalings, we can write the asymptotic normal form

$$\begin{aligned} A_{tt} &= \epsilon A - (\nu + i\Delta)A_t - |A|^2 A - zA, \\ z_t &= -\mu z + \eta|A|^2, \end{aligned} \quad (13)$$

where all coefficients are small unfolding parameters. Once again a change of variable and suitable scalings transform (13) in the complex Lorenz equations [6].

In a series of papers, Gibbon and collaborators [12] studied exhaustively the dispersive instability with small dissipation (quasireversibility): they gave conditions for the occurrence of the instability and showed using multi-scale analysis that the amplitude equations were those of the complex Lorenz model. We recover here these results with a singularity theory approach leading to normal forms and their unfoldings. We think this gives new insight in the remarkable conclusions of Gibbon. We recall that the complex Lorenz model equations can be transformed in the Maxwell-Bloch equations of the laser as shown by Haken [13]. This nice result has from our point of view a very simple interpretation since one can remark that the basic mechanism of the laser is the coupling of two oscillators in the presence of small dissipative terms and with an energy injection at zero frequency through population inversion.

We remark that Eqs. (13) with real coefficients correspond to  $(0)^2(0)$  instability with  $O(2)$  symmetry instead of

reflection symmetry. In this case the variables  $(x, y)$  of Eqs. (3) have to be replaced by complex amplitudes and the normal form will be Eqs. (13) with real coefficients due to the symmetry  $A \rightarrow \overline{A}$ . Physical examples are the Baroclinic instability with small dissipation [14] and self-focusing in the quasireversible Ginzburg-Landau equation studied by Malomed *et al.* [15].

In conclusion, we have shown that quasireversible systems present two generic codimension-one instabilities whose normal forms correspond to two well-known classical models: the real Lorenz equations and the Maxwell-Bloch equations which are then universal equations. Moreover, we propose a very simple experiment with a rotating pendulum which is by far the most simple mechanical device realizing the paradigmatic Lorenz chaos.

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