Properties of Classes of Random Graphs.

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Abstract

In [11] it is shown that the theory of almost all graphs is first order complete. Furthermore, in [3] a collection of first order axioms are given from which any first order property or its negation can be deduced. Here we show that almost all Steinhaus graphs satisfy the axioms of almost all graphs and conclude that a first order property is true for almost all graphs if and only if it is true for almost all Steinhaus graphs. We also show that certain classes of subgraphs of vertex transitive graphs are first order complete. Finally, we give a new class of higher order axioms from which it follows that large subgraphs of specified type exist in almost all graphs.

1. Introduction.

This paper is motivated by [3],[8] and [11]. In [3] and [11] it is shown that any first order property of graphs is either satisfied by almost all graphs or else almost all graphs do not satisfy the property. Furthermore, axioms are given, each of which almost all graphs satisfy, such that for any first order property either the property or its negation can be deduced from a finite number of these axioms. In fact we have an axiom for every natural number k.

Axiom k: For any set of $2k$ distinct vertices $\{v_1, v_2, v_3, \ldots, v_k, w_1, w_2, w_3, \ldots, w_k\}$, there is a vertex v with vAv_i and $\neg vAw_i$ for each $1 \leq i \leq k$.

Let us define a 0-1 matrix $(a_{i,j})_{i,j=1}^n$ as follows. Start with a 0-1 string $(a_{1,j})_{j=2}^n$. For $1 < i < j \leq n$ define $a_{i,j}$ inductively by $a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j} \pmod 2$. We complete $(a_{i,j})_{1 \leq i < j \leq n}$ to an $n \times n$ matrix by defining $a_{i,i} = 0$ and $a_{i,j} = a_{j,i}$ for $1 \leq j < i \leq n$. This gives an $n \times n$ symmetric 0-1 matrix $A = (a_{i,j})_{i,j=1}^n$ with 0s on the diagonal. The matrix A is the adjacency matrix of some graph. Any graph defined in this way is called a **Steinhaus graph**. The string $(a_{1,j})_{j=2}^n$ is called the **generating string** for the Steinhaus graph. For convenience we identify the vertex set of a Steinhaus graph with the first n natural numbers, $V_n = \{1, 2, 3, \ldots, n\}$. It is obvious that there are exactly 2^{n-1} labeled Steinhaus graphs of order n. We use the term Steinhaus graph to mean a labeled Steinhaus graph and assume that the vertex set is V_n.

We define a probability measure on Steinhaus graphs of order n by requiring that $Pr(a_{1,j} = 1) = p_{n,j}$ where $0 \leq p_{n,j} \leq 1$ for each j. We then define $q_{n,j} = 1 - p_{n,j}$ and $m_{n,j} = \min(p_{n,j}, q_{n,j})$. Any function $f(n)$ with the properties that for each sufficiently large n, $0 < f(n) < 1$ and $m_{n,j} \geq f(n)$ for each $2 \leq j \leq n$ is called a **probability bound**.

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Given a probability bound, we will say that *almost all Steinhaus graphs have property P* if the probability that a Steinhaus graph of order n has the property approaches one as n approaches infinity. Of course, the concept of "almost all" depends on the probability bound. The simplest case is the constant probability bound of $\frac{1}{2}$. This gives all Steinhaus graphs of order n the same probability.

In [8], Brigham and Dutton conjectured that almost all Steinhaus graphs have diameter two. Diameter two is a special case of Axiom 2. The conjecture was proved in [5]. It is natural to ask if all the first order axioms for almost all graphs are satisfied by Steinhaus graphs. In Section 2 we show that they are satisfied by Steinhaus graphs, which implies that almost all Steinhaus graphs have the same first order theory as almost all graphs.

In Section 3 we look at other classes of graphs whose first order theory is complete. We consider the set of all spanning subgraphs of a fixed vertex transitive graph and define a measure on that set. We show that with this measure the first order theory of almost all spanning subgraphs is complete. Furthermore, we show that with a few technical conditions on the allowed subgraphs, the theory of almost all finite subgraphs is complete and the same as the theory of almost all subgraphs with all vertices.

In Section 4 we consider higher order axioms. The first order axiom scheme described above implies that small structures exist as subgraphs of almost all graphs. The axioms introduced in Section 4 imply the existence of specified subgraphs containing all the vertices of the graph for almost all graphs. As a special case, the theorems in Section 4 imply the existence of Hamiltonian cycles in random graphs. Although the probability bounds given in the theorems are not even close to optimal in the case of Hamiltonian cycles, the theorems apply to a large class of subgraphs.

Throughout the paper we use $\log x$ to denote natural logarithm.

2. Steinhaus graphs.

A *Steinhaus triangle* is simply the adjacency matrix of a Steinhaus graph above the main diagonal. Steinhaus introduced Steinhaus triangles in [14] where he posed the problem of finding generating strings which give Steinhaus triangles with the same number of zeros as ones. This problem was solved by Harborth in [12], where he constructed such sequences in the cases where there are an even number of entries in the Steinhaus triangle.

Since then others have investigated properties of Steinhaus graphs. For example, the maximum clique size is found in [7], regular Steinhaus graphs are constructed in [1], the girth is studied in [10], and conditions under which Steinhaus graphs are bipartite are given in [9]. In [7] it is shown that the maximum diameter of a nontrivial Steinhaus graph is roughly $\frac{n}{2}$. Furthermore, in [5,6] it is shown that almost all Steinhaus graphs have diameter two. In fact, a slight modification of the proof in [5,6] shows that Axiom 1 is satisfied for almost all Steinhaus graphs. It is therefore natural to ask which of the other axioms in the axiom scheme listed above are satisfied by almost all Steinhaus graphs.

The main purpose of this section is to prove the following result.

Theorem 2.1 *Given $k \in \mathbb{N}$ and $\epsilon > 0$, almost all Steinhaus graphs with probability bound $n^{-\frac{1}{4k(2k+1)}+\epsilon}$ satisfy Axiom k.*

Before we turn to the proof of Theorem 2.1, we give some consequences of this result.

Corollary 2.2 *Given $k \in \mathbb{N}$, almost all Steinhaus graphs with constant probability bound satisfy Axiom k.*

Proof. This is obvious since if k is fixed and $0 < \epsilon < \frac{1}{4k(2k+1)}$ then $\lim_{n\to\infty} n^{-\frac{1}{4k(2k+1)}+\epsilon} = 0$. □

Corollary 2.3 *Using a constant probability bound, for any given first order property, almost all Steinhaus graphs have the property if and only if almost all graphs have the property.*

Proof. In [3] it is shown that for any first order property of graphs there is a k such that either the property or its negation can be proved with the axioms of graph theory together with Axioms 1–k. Since the same axioms hold for Steinhaus graphs, the proof of a property for almost all graphs is the same as the proof for almost all Steinhaus graphs. □

Corollary 2.3 states that random graphs look like random Steinhaus graphs from the point of view of the first order theory. Since Steinhaus graphs are easily generated, perhaps this fact could be exploited by checking algorithm performance on large random Steinhaus graphs rather than large random graphs.

Note that a proof of any true first order property of graphs requires only a finite number of the listed axioms. Given an upper bound for k, where Axiom k is required in the proof, together with Theorem 2.1 we can obtain a probability bound which implies almost all Steinhaus graphs have the property. As pointed out in [3], there is an algorithm which gives either a proof of a first order statement or a proof of its negation. Following this algorithm gives an upper bound for k.

Many higher order properties follow from first order properties. For example, diameter two (a first order property) implies connected (a second order property). Corollary 2.3 implies the following properties of almost all Steinhaus graphs: diameter two, k-connected for any fixed k, contain any fixed graph as a subgraph, not planar, and have no embedding in a surface of genus g for a fixed g.

Our proof of Theorem 2.1 is based on 5 lemmas.

Lemma 2.4 *For any fixed $2 \le k \le n$, let $(a_{i,j})$ be the adjacency matrix for a Steinhaus graph with generating string $(a_{1,j})$ given by $a_{1,r} = \delta_{k,r}$ (1 if $r = k$, 0 otherwise). Then for any $1 \le i < j \le n$, $a_{i,j} \equiv \binom{i-1}{j-k} \pmod 2$.*

Proof. This is easily verified by induction. We have inductively $a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j} \equiv \binom{i-2}{j-k-1} + \binom{i-2}{j-k} \equiv \binom{i-1}{j-k} \pmod 2$. □

Given a natural number k, denote by $H(k)$ the number of integers j with $\binom{k}{j}$ odd. For non-negative integers $a < b$ define $G(b,a) = \{k | 1 \le k \le 2^{b-1}, k = 2^a l, H(k) \le \sqrt{2^{b-a}}\}$.

Lemma 2.5 *Let a, b be non-negative integers with $a + 1 < b$. Then $|G(b,a)| \ge 2^{b-a-2}$.*

Proof. It is sufficient to show the lemma for the special case where $a = 0$ since the general case follows by dividing by 2^a. So we assume that $a = 0$.

By Lucas' Theorem [13], a binomial coefficient $\binom{k}{m}$ is odd if and only if for each w, if there is a zero in the base two representation of k in the 2^w place then there is a zero in the 2^w place in the base two representation of m. Therefore, $k \in G(b, 0)$ if and only if the number of ones in the base two representation of k is no more than $\frac{b}{2}$. Obviously there are at least $\frac{1}{2}2^{b-1} = 2^{b-2}$ such values for k. □

For any pair of vertices (v, w) in a Steinhaus graph, we define $C(v, w) = |v - w| + 1$. Furthermore, for any subset $T \subset V_n$ and any $v \in V_n$ we let $C_T(v) = \{C(v, w)|w \in T\}$. By Lemma 2.4, given any Steinhaus graph one can change the adjacency of vertices v and w simply by changing the value of the generating string at position $C(v, w)$. Furthermore, this does not change the adjacency of any pair of vertices (u, z) with $C(u, z) > C(v, w)$. We let $B(v, w) = \{i|\binom{\min\{v,w\}-1}{\max\{v,w\}-i} \equiv 1 \pmod 2\}$. Then we let $B_T(v) = \bigcup_{w \in T} B(v, w)$. By Lemma 2.4, $B_T(v)$ is the set of all entries in the generating string of a Steinhaus graph which when changed will change the (v, w) entry in the adjacency matrix for some $w \in T$.

Let $T \subset V_n$. We say a sequence $v_1, v_2, \ldots, v_r \in V_n$ is **T-independent** if for each $1 \le i \le r$, $C_T(v_i) \cap B_T(v_j) = \emptyset$ for $j < i$ and $|C_T(v_i)| = |T|$ for each i.

Lemma 2.6 *Suppose $v_1, \ldots, v_r$ is a T-independent sequence for some set $T \subset V_n$. Then for any generating string for a Steinhaus graph with n vertices, by changing only the entries in the generating string indexed by $C_T(v_i)$ it is possible to attain any combination of adjacencies between v_i and the vertices in T. Furthermore, making these changes does not change the adjacencies between v_j and the vertices of T for any $j < i$.*

Proof. Let $(a_{1,j})$ be an arbitrary generating string for a Steinhaus graph which gives an adjacency matrix $(a_{i,j})$. Label the elements in T by $w_1, \ldots, w_t$ where $C(v_i, w_1) > \cdots > C(v_i, w_t)$. It is then clear that by changing the value of $a_{1,C(v_i,w_r)}$, the value a_{v_i,w_r} changes. Furthermore, changing $a_{1,C(v_i,w_r)}$ does not change the value of a_{v_i,w_k} for $k < r$, nor does it change the change the value of a_{v_j,w_s} for any $j < i$ and any s. □

Lemma 2.7 *Let $T = \{v_1, \ldots, v_k, w_1, \ldots, w_k\} \subset V_n$ and suppose that m_n is the probability bound used for Steinhaus graphs. If there is a T-independent sequence of length r then the probability that Axiom k is true for a Steinhaus graph using the set T is at least $1 - (1 - m_n^{2k})^r$.*

Proof. Let $z_1, \ldots, z_r$ be a T-independent sequence of length r. Note that the generating strings for Steinhaus graphs for which Axiom k fails for $z_1, \ldots, z_{j-1}$ using T, can be partitioned into subsets each of size 2^{2k} by putting two strings in the same subset if the strings agree in every entry except for positions $(1, i)$ where $i \in C_T(z_j)$. By Lemma 2.6, in each subset there is a sequence whose Steinhaus graph satisfies Axiom k using T and z_j. Therefore, the probability that a Steinhaus graph satisfies Axiom k using z_j and T given that it does not satisfy Axiom k using T and any z_i with $i < j$ is at least m_n^{2k}. It is then clear that a lower bound that Axiom k is satisfied using T is $1 - (1 - m_n^{2k})^r$. □

The goal, based on Lemma 2.7, is to find a long T-independent sequence. The construction of a long T-independent sequence is sufficient (although perhaps not necessary) to prove Theorem 2.1.

Lemma 2.8 *Given $T \subset V_n$ with $|T| = 2k$, there is a T-independent sequence of length at least* $\frac{n^{\frac{1}{2(2k+1)}}}{32k^2} - 1$.

Proof. Let $v_1, \ldots, v_{2k}$ be the elements of T listed in order and write each as $v_i = n^{\alpha_i}$. Let $\alpha_0 = 0$ and $\alpha_{2k+1} = 1$ Pick $0 \leq i \leq 2k$ so that $\alpha_{i+1} - \alpha_i \geq \frac{1}{2k+1}$. Certainly there is at least one such i since the α_is break the unit interval into $2k+1$ intervals. Now, let a and b be such that $v_i < 2^a$ and $2^b < v_{i+1}$ with $b-a$ as large as possible. If $b \leq a+1$ then $n^{\frac{1}{2k+1}} < 8$, so the lemma is obviously true. We therefore assume $b > a+1$. Then $2^{b-a} \geq \frac{1}{4} n^{\frac{1}{2k+1}}$. Let $G = G(b,a)$. By Lemma 2.5 $|G| \geq 2^{b-a-2}$. List the elements in G in order and label them $h_1, h_2, \ldots$. We start a T-independent set by setting $x_1 = h_i$ where i is as small as possible with $\{h_i\}$ T-independent. We then assume inductively that $S_r = \{x_1, \ldots, x_r\} \subset G$ is a T-independent set and attempt to add another element $x_{r+1} \in G$ to S by always choosing x_{r+1} the smallest element in G so that $S_{r+1} = S_r \cup \{x_{r+1}\}$ is T-independent.

There are two required conditions for S_{r+1} to be T-independent. First, $|C_T(x_{r+1})| = |T|$. This condition fails for at most $|T|(|T|-1) = 2k(2k-1)$ vertices. The next condition is that $C_T(x_{r+1}) \cap B_T(x_i) = \emptyset$ for $i \leq r$. So, suppose that $w_1, w_2 \in T$ and $C(w_1, x_{r+1}) \in B(x_i, w_2)$. Every element of T is either less than 2^a or else it is larger than 2^b. This gives four cases based on the values of w_1 and w_2. It is easily checked that $C(x_{r+1}, w_1) \notin B(x_i, w_2)$ for either w_1 or w_2 less than 2^a. So suppose that $w_1, w_2 > 2^b$. Note that $C(w_1, x) = w_1 - x + 1$ is a strictly decreasing function of x for $x < w_1$. Furthermore, there are at most $r(2k)2^{\frac{b-a}{2}}$ forbidden values for $C(w_1, v_{r+1})$. Therefore as long as $2^{b-a-2} - (2k)r(2k)2^{\frac{b-a}{2}} - 2k(2k-1) \geq 1$ there is at least one vertex $v_{r+1} \in G$ so that $S_{r+1} = S_r \cup \{v_{r+1}\}$ is T-independent. Solving the inequality gives $r < \frac{2^{b-a-2}}{(2k)^2 2^{\frac{b-a}{2}}} - \frac{2k-1}{(2k)2^{\frac{b-a}{2}}} - \frac{1}{(2k)^2(2^{\frac{b-a}{2}})} = N$. Note that $N > \frac{n^{\frac{1}{2(2k+1)}}}{32k^2} - 1$, which completes the proof. □

Proof of Theorem 2.1. Using Lemmas 2.8 and 2.7, we have that the probability that Axiom k fails is no more than

$$P_n = \binom{n}{k}\binom{n-k}{k}\left(1 - m_n^{2k}\right)^{\frac{n^{\frac{1}{2(2k+1)}}}{32k^2} - 1}.$$

Taking logarithms and substituting $m_n > n^{-\frac{1}{4k(2k+1)}+\epsilon}$ it is easily shown that $P_n \to 0$. □

Note that if we use a constant probability bound $m_n = m$ then the above estimate of the probability that Axiom k fails is roughly order $c^{an^{\frac{1}{2(2k+1)}}+\epsilon}$ where $c = 1 - m^{2k}$ and $a = \frac{1}{32k^2}$. In the special case of $m = \frac{1}{2}$ (which means each entry in the generating string has probability $\frac{1}{2}$ of being a 1), the probability bound can be improved to roughly $c^{\frac{n}{8k^2}}$ by using different arguments which are outlined below. Unfortunately, these arguments fail for any $m < \frac{1}{2}$. It seems that for m near $\frac{1}{2}$ it should be possible to improve the bound given above to make it closer to the better bound in the case where $m = \frac{1}{2}$.

Theorem 2.9 *The ratio of the number of labeled Steinhaus graphs of order n which do not satisfy Axiom k to the total number of labeled Steinhaus graphs of order n is no more* *than* $\binom{n}{k}\binom{n-k}{k}\left(\frac{2^{2k}-1}{2^{2k}}\right)^{\frac{n-10k^2+k-1}{8k^2}}$. ◻

We do not prove Theorem 2.9, but simply give the essential difference between the proof of Theorem 2.9 and Theorem 2.1. Instead of looking for a large T-independent set, we look for a large set $\{v_1, \ldots, v_r\}$ with the property that the sets $C_T(v_i)$ are disjoint and $|C_T(v_i)| = |T|$. Obviously it is easier to find large sets with this property than it is to find large T-independent sets. In fact, it is not difficult to show there are sets of size at least $\frac{n-10k^2+k-1}{8k^2}$ with the stated property. In either proof the values in $C_T(v_i)$ are allowed to change in order to estimate a lower bound for the probability that Axiom k is satisfied by v_i and T. In Lemma 2.7 it was necessary to assume that the set $v_1, \ldots, v_r$ is T-independent. The difference in the case $m = \frac{1}{2}$ is that all the combinations of ones and zeros in $C_T(v_i)$ are equally likely. This gives independence of the events that Axiom k is satisfied by T and v_i for each of the values of i in the set.

3. Completeness in Vertex Transitive Graphs.

In this section we show that the first order theories of some families of graphs are complete in the "almost everywhere" sense, although different from the first order theory of random graphs.

To begin we say that M is a **master graph** if M is a graph with a countable number of vertices $v_1, v_2, v_3, \ldots$. We consider the collection Γ of all spanning subgraphs G of M. Let $E(M)$ denote the edge set of M. We make Γ a probability space by letting the probability of any edge in M be p. More precisely, we have a map $\theta : 2^{E(M)} \to \Gamma$ given by $\theta(f) = G$ where (v_i, v_j) is an edge in G if and only if $f(v_i, v_j) = 1$. The measure on Γ is then just the image under θ of the Bernoulli measure on $2^{E(M)}$ with parameter p.

Here we let M be a vertex transitive graph with a countably infinite vertex set and with every vertex having finite degree.

Given vertices $v_1, v_2 \ldots, v_n \in M$ and $N \in \mathbb{N}$, the N component $C_N^G(v_1, \ldots, v_n)$ of $(v_1, v_2, \ldots, v_n)$ in G is defined to be the set of all vertices $v \in G$ such that $\rho(v, v_i) \leq N$ for some $1 \leq i \leq n$, where ρ is the usual edge distance in G. We will think of $C_N^G(v_1, \ldots, v_n)$ as either a set of vertices or else the subgraph induced by the set of vertices. We take $\rho(w, z) = \infty$ if w and z are in different components. Next we let $C_N^G(v_1, \ldots, v_n) = S_N^1(v_1, \ldots, v_n) \cup \cdots \cup S_N^m(v_1, \ldots, v_n)$ where each $S_N^i(v_1, \ldots, v_n)$ is a connected component of $C_n^G(v_1, \ldots, v_n)$. Note that each $S_N^i(v_1, \ldots, v_n)$ contains at least one of the vertices $v_1, \ldots, v_n$, and the diameter of each $S_N^i(v_1, \ldots, v_n)$ is not more than $2nN$ for all i.

Let $\tilde{M}$ be an isomorphic copy of M. Suppose that $T_1, \ldots, T_m$ is a collection of finite subgraphs of $\tilde{M}$, each containing a fixed vertex $\tilde{v}_0$. Furthermore, suppose for each $1 \leq i \leq m$ we have a subset B_i of $\{x_1, x_2, \ldots, x_n\}$ and a function $\varphi_i : B_i \to V(T_i)$ such that

(1) $\bigcup_i B_i = \{x_1, x_2, \ldots, x_n\}$ and the B_i are pairwise disjoint.

(2) each T_i is connected, and for each $v_j \in T_i$, $\rho(\varphi(B_i), v_j) \leq N$. Also, $\tilde{v}_0 \in \varphi(B_i)$.

We call the collection $T_1, \ldots, T_m$, together with the corresponding functions $\varphi_1, \ldots, \varphi_m$, an (n, N)-**configuration** $\mathcal{C}_{n,N}$. Occasionally we shall refer to the vertex $\varphi_j(x_i)$ as the *vertex identified with* x_i, and the set of all the vertices identified with some x_i as the *identified vertices*.

The purpose of defining (n, N)-configurations is to translate logical formulae into configurations. Roughly speaking, for any formula $\phi(x_1, \ldots, x_n)$ there is an N so that for almost all graphs and all choices of $v_1, \ldots, v_n$ the truth of $\phi(v_1, \ldots, v_n)$ is determined by the induced subgraph on the vertex set $C_N^G(v_1, \ldots, v_n)$. A key point is that any possible finite subgraph of M should normally appear infinitely often in a random graph from Γ.

Since we assume that M has only finite degree, there are only finitely many $\mathcal{C}_{n,N}$ for each n and N. We say that $\mathcal{C}_{n,N} = T_1 \cup \cdots \cup T_m$ and $\mathcal{C}'_{n,N} = T'_1 \cup \cdots \cup T'_{m'}$ are isomorphic if $m = m'$ and for each i there is a graph isomorphism $\psi : T_i \to T'_i$ such that $\psi \circ \varphi_i(x_j) = \varphi'_i(x_j)$ for each $j \in B_i$.

Given $G \in \Gamma$ and vertices $v_1, \ldots, v_n$ in G, we say that $(v_1, \ldots, v_n)$ **satisfies** $\mathcal{C}_{n,N} = T_1 \cup \cdots \cup T_m$ if $C_N(v_1, \ldots, v_n) = S_N^1(v_1, \ldots, v_n) \cup \cdots \cup S_N^m(v_1, \ldots, v_n)$ and for all $1 \le i \le m$ there is a graph isomorphism $\psi_i : S_N^i(v_1, \ldots, v_m) \to T_i$ such that $\psi_i(v_j) = \varphi_i(x_j)$.

We say that a graph $G \in \Gamma$ is **good** if for any finite connected subgraph S of M containing v_0, there are infinitely many $v \in V(G)$ such that $C_N(v)$ is isomorphic with S by an isomorphism sending v to $\tilde{v}_0$. Clearly the set of good graphs in Γ has probability measure 1.

We consider finite sets of (n, N)-configurations, $\mathcal{D}_{n,N} = \{\mathcal{C}_{n,N}^1, \ldots, \mathcal{C}_{n,N}^k\}$ for fixed n and N. For any $G \in \Gamma$ and formula $\phi(x_1, \ldots, x_n)$ we say ϕ is **equivalent** to $\mathcal{D}_{n,N}$ if first for every $(v_1, \ldots, v_n)$, vertices in G, $G \models \phi(v_1, \ldots, v_n)$ if and only if some $\mathcal{C}_{n,N}^j \in \mathcal{D}_{n,N}$ satisfies $(v_1, \ldots, v_n)$ in G. We further require that the set $\mathcal{D}_{n,N}$ be maximal in the sense that if $G \models \phi(v_1, \ldots, v_n)$ and $\mathcal{C}_{n,N}$ satisfies $(v_1, \ldots, v_n)$ then $\mathcal{C}_{n,N} \in \mathcal{D}_{n,N}$. Note that $\mathcal{D}_{n,N}$ is closed under isomorphism, that is, if $\mathcal{C}_{n,N}, \mathcal{C}'_{n,N} \in \mathcal{D}_{n,N}$ are isomorphic and $\mathcal{C}_{n,N} \in \mathcal{D}_{n,N}$ then $\mathcal{C}'_{n,N} \in \mathcal{D}_{n,N}$. This follows since $\mathcal{D}_{n,N}$ is required to be maximal. Thus, we may view $\mathcal{D}_{n,N}$ as a set of isomorphism classes of (n, N)-configurations.

If $N < N'$, and $\mathcal{C}'_{n,N'}$, $\mathcal{C}_{n,N}$ are configurations we say that $\mathcal{C}_{n,N}$ is the restriction of $\mathcal{C}'_{n,N'}$ or that $\mathcal{C}'_{n,N'}$ is an extension of $\mathcal{C}_{n,N}$ if the configuration obtained from $\mathcal{C}'_{n,N'}$ by taking the induced subgraphs of $\mathcal{C}'_{n,N'}$ consisting of all vertices within a distance of N of at least one of the identified vertices is isomorphic with $\mathcal{C}_{n,N}$. For a fixed set of configurations $\mathcal{D}_{n,N}$ and $N < N'$ we form the N'-extension of $\mathcal{D}_{n,N}$ by simply including all $C'_{n,N'}$ which extend some $\mathcal{C}_{n,N} \in \mathcal{D}_{n,N}$. It is easy to see that if $\phi(x_1, \ldots, x_n)$ is equivalent to $\mathcal{D}_{n,N}$ then $\phi(x_1, \ldots, x_n)$ is equivalent to any extension of $\mathcal{D}_{n,N}$.

Theorem 3.1 *Let M be a vertex transitive graph of finite degree and Γ the corresponding family of graphs. For each formula $\phi(x_1, \ldots, x_n)$ there is an $N \in \mathbb{N}$ and $\mathcal{D}_{n,N}$, a set of isomorphism classes of (n, N)-configurations, such that for all good $G \in \Gamma$, ϕ is equivalent to $\mathcal{D}_{n,N}$ for G.*

Proof. We first give a procedure to construct $\mathcal{D}_{n,N}$ which we then show to be equivalent to ϕ. We inductively build up the configurations.

As a start assume that ϕ is atomic. Suppose $\phi(x_1, \ldots, x_n) = x_i A x_j$. We take $N = 1$ and let $\mathcal{D}_{n,1}$ be the set of $(n, 1)$-configurations $\mathcal{C}_{n,1}$ for which x_i, x_j are identified with vertices $\tilde{v}_i, \tilde{v}_j$ which are connected by an edge in $\mathcal{C}_{n,1}$. It is clear that ϕ is equivalent to $\mathcal{D}_{n,1}$. For the other atomic case, $x_i = x_j$ it is clear that we should take $N = 0$ and $\mathcal{D}_{n,0}$ to be the set of $(n, 0)$-configurations $\mathcal{C}_{n,0}$ for which the vertices identified with x_i and x_j are equal.

Suppose the theorem holds for formulas ϕ and ψ giving equivalent configurations $\mathcal{D}_{n,N}$ and $\mathcal{D}'_{n,N'}$ respectively. It is easy to find the appropriate sets of configurations for the boolean combinations of ϕ and ψ. In particular, for $\neg\phi$ we simply take all $\mathcal{C}_{n,N}$ for which $\mathcal{C}_{n,N} \notin \mathcal{D}_{n,N}$ where ϕ is equivalent to $\mathcal{D}_{n,N}$. In the other two cases we simply take the intersection or the union of the equivalent configurations after forming the appropriate extensions so the N parameters are equal. Note that in the Boolean case we may increase the value of N for one of the configurations but it is only by a simple extension.

Assume now without loss of generality that $\phi(x_1, \ldots, x_n) = \exists x_{n+1}\psi(x_1, \ldots, x_n, x_{n+1})$, and let $\mathcal{D}_{n+1,N} = (\mathcal{C}^1_{n+1,N}, \ldots, \mathcal{C}^k_{n+1,N})$ be equivalent to ψ. We let $\tilde{\mathcal{D}}_{n,3N}$ be the set of $\mathcal{C}_{n,3N}$ satisfying one of the following:

1. $\mathcal{C}_{n,3N} = T_1 \cup \cdots \cup T_m$ and there is a vertex $\tilde{v} \in T_i$ for some $1 \leq i \leq m$ such that $\tilde{v}$ is within $2N$ of $\tilde{v}_r$ for some r and for some $1 \leq j' \leq k$ $\mathcal{C}^{j'}_{n+1,N}$ is obtained from $\mathcal{C}_{n,3N}$ by identifying x_{n+1} with $\tilde{v}$ and then "restricting to N". That is, $\mathcal{C}^{j'}_{n+1,N} = C^{\mathcal{C}_{n,3N}}_N(\tilde{v}_1, \ldots, \tilde{v}_n, \tilde{v})$ where $x_1, \ldots, x_n$ are identified with $\tilde{v}_1, \ldots, \tilde{v}_n$ in $\mathcal{C}_{n,3N}$.

2. For some $1 \leq j' \leq k$, $\mathcal{C}^{j'}_{n+1,N}$ is such that $\tilde{v}_{n+1}$ is the only identified vertex in some component of $\mathcal{C}^{j'}_{n+1,N}$ and $\mathcal{C}_{n,3N}$ is a $3N$ extension of the remaining components of $C^{j'}_{n+1,N}$.

We show that $\tilde{\mathcal{D}}_{n,3N}$ is equivalent to $\phi(x_1, \ldots, x_n)$. Let $G \in \Gamma$ be good, and let $v_1, \ldots, v_n \in G$. Assume first that $G \models \phi(v_1, \ldots, v_n)$, so let $v_{n+1} \in G$ be such that $G \models \psi(v_1, \ldots, v_n, v_{n+1})$. Consider $C = C^G_N(v_1, \ldots, v_n, v_{n+1})$.

First assume that v_{n+1} is the only vertex among $(v_1, \ldots, v_n, v_{n+1})$ in the component of C containing v_{n+1}. Let $\mathcal{C}_{n+1,N}$ be an (n, N)-configuration satisfying $v_1, \ldots, v_{n+1}$. Say $\mathcal{C}_{n+1,N} = T_1 \cup \cdots \cup T_m$ where T_m is the component containing $\tilde{v}_{n+1}$. By induction $\mathcal{C}_{n+1,N} \in \mathcal{D}_{n+1,N}$. If $\mathcal{C}_{n,3N}$ is a $3N$-configuration realizing $(v_1, \ldots, v_n)$ in G, then clearly $\mathcal{C}_{n,3N} \in \tilde{\mathcal{D}}_{n,3N}$ from Case 2.

Assume next that each component of C contains an element of $\{v_1, \ldots, v_n\}$. It follows that $v_{n+1} \in C^G_{2N}(v_1, \ldots, v_n)$. Let $\mathcal{C}_{n,3N}$ again be the $3N$-configuration realizing $(v_1, \ldots, v_n)$ in G. Let $\tilde{v}_{n+1}$ be the vertex corresponding to v_{n+1} under a fixed isomorphism of $\mathcal{C}_{n,3N}$ with $C^G_{3N}(v_1, \ldots, v_n)$. Note that the restriction of $\mathcal{C}_{n,3N}$ to the components of $C^{\mathcal{C}_{n,3N}}_N(\tilde{v}_1, \ldots, \tilde{v}_n, \tilde{v}_{n+1})$ is isomorphic to $\mathcal{C}_{n+1,N}$, an (n, N)-configuration realizing $v_1, \ldots, v_n, v_{n+1}$. By induction, $\mathcal{C}_{n+1,N} \in \mathcal{D}_{n+1,N}$. Hence $\mathcal{C}_{n,3N} \in \tilde{\mathcal{D}}_{n,3N}$ by virtue of Case 1.

Next suppose $v_1, \ldots, v_n \in G$ are such that some $\tilde{\mathcal{C}}_{n,3N} \in \tilde{\mathcal{D}}_{n,3N}$ realize $v_1, \ldots, v_n$. Suppose first that $\tilde{\mathcal{C}}_{n,3N} \in \tilde{\mathcal{D}}_{n,3N}$ by virtue of Case 1. Fix an isomorphism π between $\tilde{\mathcal{C}}_{n,3N}$ and $C^G_{3N}(v_1, \ldots, v_n)$. So $\pi(\tilde{v}_i) = v_i$ for $1 \leq i \leq n$. Let $\tilde{v}_{n+1}$ be the vertex in $\tilde{\mathcal{C}}_{n,3N}$ as in Case 1, and let $v_{n+1} = \pi(\tilde{v}_{n+1})$. Thus, if $\mathcal{C}_{n+1,N}$ is the $(n+1, N)$-configuration obtained from $\tilde{\mathcal{C}}_{n,3N}$ by restricting $\tilde{\mathcal{C}}_{n,3N}$ to $C^{\mathcal{C}_{n,3N}}_N(\tilde{v}_1, \ldots, \tilde{v}_n, \tilde{v}_{n+1})$ then $\mathcal{C}_{n+1,N} \in \mathcal{D}_{n+1,N}$ and so by induction $G \models \psi(v_1, \ldots, v_n, v_{n+1})$, and hence $G \models \phi(v_1, \ldots, v_n)$. Note that we have not yet used the fact that G is good.

Suppose now that $\tilde{\mathcal{C}}_{n,3N} \in \tilde{\mathcal{D}}_{n,3N}$ by virtue of Case 2. Let $\mathcal{C}_{n+1,N} \in \mathcal{D}_{n+1,N}$ be as in Case 2. Let $\mathcal{C}_{n+1,N} = T_1 \cup \cdots \cup T_m$ and assume T_m is the component of $\mathcal{C}_{n+1,N}$ with $\tilde{v}_{n+1}$ the only identified vertex. Since G is good, let $v_{n+1} \in G$ be such that $v_{n+1} \notin C^G_{2N}(v_1, \ldots, v_n)$ and $C^G_N(v_{n+1})$ is isomorphic with T_m. However, then $\mathcal{C}_{n+1,N}$ satisfies $(v_1, \ldots, v_n, v_{n+1})$. So, by induction $G \models \psi(v_1, \ldots, v_n, v_{n+1})$ and hence $G \models \phi(v_1, \ldots, v_n)$. □

If $n = 0$ in the above argument then ϕ is equivalent to $\top$ or $-$ depending on whether $\mathcal{D}_{n+1,N}$ is non-empty or not. Hence Theorem 3.1 has the following consequences

Corollary 3.2 *Let M be a vertex transitive graph of finite degree and let Γ be the corresponding class of random subgraphs. Then for any sentence ϕ in the first order language of graph theory $\mu\{G \in \Gamma | G \models \phi\} \in \{0, 1\}$.*

A closer examination of the argument above allows us to obtain more. The fact that our reduction worked on first order formulae (not just sentences) yields the following assertion.

Corollary 3.3 *With M and Γ as above, $\phi(x_1, \ldots, x_n)$ a first order formula, and $v_1, \ldots, v_n \in M$, let $\alpha(v_1, \ldots, v_n) = \mu\{G \in \Gamma | G \models \phi(v_1, \ldots, v_n)\}$. Then the set of all possible values for α, $\{\alpha(v_1, \ldots, v_n) | v_1, \ldots, v_n \in M\}$, is finite.*

Proof. This follows from the theorem noting that $\phi(x_1, \ldots, x_n)$ is equivalent to some $\mathcal{D}_{n,N}$ for some $N \in \mathbb{N}$. Thus, on the measure one set of good G, whether $G \models \phi(v_1, \ldots, v_n)$ depends only on $C^G_N(v_1, \ldots, v_n)$. Since M has finite degree, the probability measure can be restricted to a finite probability space. □

Given the vertex transitive master graph M of finite degree, we say a sequence $M_1 \subset M_2 \subset \cdots \subset M_i \subset \cdots$ with $M_i \subset M$ is a reduction of M if the following three conditions are satisfied.

1. Each M_i is a finite spanning subgraph of M.
2. Let $r^*(M_i) = \sup_{v \in M_i}\{\sup\{n \in \mathbb{N} | \rho(v, v') \leq n \text{ implies } v' \in M_i\}\}$. (So $r^*(M_i)$ is the radius of the largest ball in M_i.) Then for every $c > 0$, $r^*(M_i) > c\log(i)$ for all i sufficiently large. This is saying that M_i contains a ball of radius greater than $c\log(i)$.
3. For some D, $l > 0$, M_i contains no more than Di^l vertices.

As an illustration, let $\mathbb{Z}$ represent the graph whose vertex set is the set of integers and edges consist of consecutive integers. If we let $H = \mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ we may take M_i to be the induced subgraph on the vertices with all coordinates between $-i$ and i. We are then considering random subgraphs of the lattice graph on $(-i, i) \times \cdots \times (-i, i)$. It is easily seen that this forms a reduction of H.

Each M_i determines a finite probability space using the Bernoulli measure in the usual manner with probability of edge connection p. We say that almost all finite spanning subgraphs (relative to the reduction) satisfy ϕ if $\lim_{i\to\infty} P(i) = 1$, where $P(i)$ is the probability of a graph in M_i satisfying ϕ.

Theorem 3.4 *Let M be a vertex transitive graph of finite degree and let $M_1 \subset M_2 \subset \cdots \subset M_i \subset \cdots$ be a reduction for M. Then for any sentence ϕ in the first order theory* *of graph theory, either almost all graphs (relative to the reduction) satisfy ϕ or almost all satisfy $\neg\phi$.*

Proof. Let $G \subset M_i$. For fixed n, $N \in \mathbb{N}$ we say G is n, N good if for all vertices $v_1, \dots v_n \in G$ and T a connected subset of $C_N^M(\tilde{v}_0)$ there exists $v_{n+1} \in G$ with $v_{n+1} \notin C_{2N}^G(v_1, \dots, v_n)$ and $C_N^G(v_n)$ is isomorphic to T by an isomorphism sending v_n to $\tilde{v}_0$.

Investigating the proof of Theorem 3.1 shows that it is sufficient to show that almost all G (relative to M_i) are n, N good, for any fixed n, N.

We say G is N good relative to $v_1, \dots, v_n$ if there is a $v_{n+1} \in G$ as above for the vertices $v_1, \dots, v_n$. Let $b(v_1, \dots, v_n)$ represent the probability that G is not good relative to $v_1, \dots, v_n$. For fixed i and $G \subset M_i$, the probabilities that G is not n, N good is at most $\Sigma_{v_1,\dots,v_n} b(v_1, \dots, v_n) \leq \binom{Di^l}{n} b(v_1, \dots, v_n)$ for some $v_1, \dots, v_n$.

Fix $v_1, \dots, v_n \in G$. Since $r^*(M_i) > c\log(i)$ (c sufficiently large) there are at least $\frac{c\log(i)}{6N} - n$ vertices $v \in G$ at least $2N+1$ apart such that $C_N^G(v) \subset G$ and $v \notin C_{2N}^G(v_1, \dots, v_n)$. The probabilities that $C_N^G(v)$ is isomorphic to T, for a fixed T, are independent for the $\frac{c\log(i)}{6N} - n$ different vertices. The possible number of Ts is a function only of N, say $w(N)$. We divide the vertices v above into $w(N)$ sets of (roughly) equal size, and within each set attempt to find at least one v with $C_N^G(v)$ isomorphic to the corresponding T. The probability we do not succeed is at most

$$w(N)(1-q)^{\left(\frac{c\log(i)}{6N\,w(N)} - \frac{n}{w(N)}\right)}$$

for some $q > 0$. Thus, the probability G is n, N good is at least

$$1 - \binom{Di^l}{n} w(N)(1-q)^{\left(\frac{c\log(i)}{6N\,w(N)} - \frac{n}{w(N)}\right)}.$$

Since c can be taken large compared to N and l, the above probability approaches one. □

It is interesting to note that in the infinite and finite cases one has the same reduction procedure of ϕ to $\top$ or $-$. In particular, the infinite and the finite cases have the same first order theory as long as the three conditions are met in the finite case. An example of this is the following assertion.

Corollary 3.5 *For any first order formula ϕ in the language of graph theory, almost all spanning subgraphs of $\mathbb{Z}^n$ satisfy ϕ if and only if almost all spanning subgraphs of $(-N, N)^n$ satisfy ϕ.*

Finally we wish to point out that the fact that the master graph has finite degrees is crucial to our argument. It would be interesting to see a proof of Corollary 3.2 which would include the case of infinite degree master graphs. Of course, this proof would then include Corollary 3.2 as well as the fact that the almost all theory of graphs is complete (in this case the master graph is simply the complete graph on the integers).

4. Higher order axioms.

We consider now some higher order properties of graphs which generalize the first order axioms given in Section 1. The axioms of Section 1 imply the existence of "local" structures. For example, for any fixed graph almost all graphs have a subgraph isomorphic to the fixed graph. The axioms of this section imply the existence of "global" structures, such as hamiltonian cycles in almost all graphs. The global structures are constructed according to a "pattern", which is made precise in the definition of a constructing family. We prove two theorems in this section. Theorem 4.1 asserts the existence of large subgraphs and furthermore allows negative edge requirements. In Theorem 4.4 we only assert the existence of large subgraphs but with improved estimates.

A **constructing family of functions** $\Psi = \{F_n^{(k,l)}\}$ **of type** (k,l) is a collection of funtions $F_n^{(k,l)}$ for each $n \in \mathbb{N}$ such that

1) $F_n^{(k,l)}$ has domain tuples $(S_0, <_{S_0}, \dots, S_j, <_{S_j}, (x_1, \dots, x_k), (y_1 \dots, y_l))$ where $S_i \subset \{1, \dots, n\}$ with $|S_i| < n$ for each i, $<_{S_i}$ is a linear ordering of a subset of S_i for each i, $x_1, \dots, x_k \in S_j$ are distinct, and $y_1, \dots, y_l \in \{1, 2, \dots, n\} - S_j$ are distinct,
2) $F_n^{(k,l)}(S_0, <_{S_0}, \dots, S_j, <_{S_j}, (x_1, \dots, x_k), (y_1, \dots, y_l)) = (S_{j+1}, <_{S_{j+1}})$, where $S_{j+1} = S_j \cup \{y_1, \dots, y_l\}$, and $<_{S_{j+1}}$ is a linear ordering of a subset of S_{j+1}.

The intended meaning of a constructing family is to give a procedure for constructing a large subgraph according to a specific pattern. For example, to construct a hamiltonian cycle we start with a small cycle and add one vertex at each step. The vertex is chosen to be adjacent with consecutive vertices around the cycle, increasing the length of the cycle by one. This corresponds to a construction family of type $(2,1)$. In particular, S_i is the set of vertices in the cycle in step i while $<_{S_i}$ is the order around the cycle starting at any fixed vertex. The set S_{i+1} is the new cycle with the added vertex and $<_{S_{i+1}}$ is again the order around the newly formed cycle. Note in this case $F_n^{(k,l)}(S_0, <_{S_0}, \dots, S_j, <_{S_j}, (x_1, x_2), (y_1))$ depends only on $(S_j, <_{S_j}, (x_1, x_2), (y_1))$.

We say that the functions $f_n : \mathbb{N} \to \mathbb{R}$ are **bounding functions** for $\{F_n^{(k,l)}\}$ if, with the above notation, $|\text{field}(<_{S_j})| \geq f_n(|S_j|)$. Here field refers to the set of elements x where either $x <_{S_j} y$ or else $y <_{S_j} x$ for some y.

The graph in Figure 2a can also be constructed with a $(2,1)$ family of functions. In this graph we have approximately $n^{1/16}$ paths of length $n^{1/16}$ which we will refer to as the cycles. We require that the end vertices of each path are not connected (the dotted line in the figure), they are connected as shown, and have attached paths of lengths approximately $n^{7/8}$ as shown. In the first $n^{3/16}$ steps we build up a subgraph consisting of all the cycles and attached paths of length $n^{1/16}$. For j in this range we take the field of $<_{S_j}$ to consist of only two vertices. For $j \geq n^{3/16}$ the constructing function adds at each step a single vertex to one of the paths. At each step the path chosen is the next one clockwise from the previous one. The field of S_{j+1} is simply the next path to be added to. The order $<_{S_{j+1}}$ is the order along this path. For $j > n^{3/16}$ the size of the field of S_j is at least $|S_j|^{1/3}$.

We consider the following axiom scheme for a graph G with n vertices. The axioms say that the construction procedure corresponding to $F_n^{k,l}$ can be carried out in G.

Axiom $\mathbf{A^{(k,l)}}$: Let d be a graph with vertex set $\{a_1 \dots, a_k, b_1, \dots b_l\}$ and let d_{in} be a graph with vertex set $\{c_1, \dots, c_r\}$ where $k \leq r \leq k+l$. There exists a sequence $(S_0, <_{S_0})$, $\dots$, $(S_m, <_{S_m})$ with $m = \left[\frac{n-r}{l}\right]$, where S_i is a subset of the vertices of G and

1) the subgraph of G induced from the vertices S_0 is isomorphic with d_{in}. Also, the ordering $<_{S_0}$ on S_0 corresponds to an ordering $c_1 < c_2 < \dots < c_r$ under the isomor-

phism;

2) for every $0 \le j \le m-1$, $F_n^{(k,l)}(S_0, <_{S_0}, \ldots, S_j, <_{S_j}, (x_1, \ldots, x_k), (y_1 \ldots, y_l)) = (S_{j+1}, <_{S_{j+1}}$) for some elements $x_1, \ldots, x_k \in S_j$, $y_1, \ldots, y_l \notin S_j$, and where $x_1, \ldots, x_k$ are consecutive elements of S_j with respect to $<_{S_j}$; also, $x_i A y_s$ in G if and only if $a_i A b_s$ in d and $y_i A y_s$ in G if and only if $b_i A b_s$ in d;

3) if $e = n - \left(r + l\left[\frac{n-r}{l}\right]\right) > 0$, then there are $y_1, \ldots, y_e$, vertices in G which are not in S_m, and consecutive elements (using order $<_{S_m}$) $x_1, \ldots, x_k$ such that $x_i A y_s$ in G if and only if $a_i A b_s$ in d and $y_i A y_s$ in G if and only if $b_i A b_s$ in d.

If the vertices $x_1, \ldots, x_k, y_1, \ldots, y_w$ (for $w = l$ or $w = e$) satisfy Property 2 or Property 3 above then we say that $x_1, \ldots, x_k, y_1, \ldots, y_w$ **fit** d.

Note that Axiom (k, l) has $F_n^{(k,l)}$ as a parameter in its statement. Given a constructing family $\{F_n^{(k,l)}\}$, it makes sense to ask if a graph G satisfies Axiom (k, l). We will show below that for $\{F_n^{(k,l)}\}$ having suitable bounding functions $f(n)$, for each k, l and d, d_{in} Axiom (k, l) holds for "almost all graphs". We interpret almost all as in Bollobás [4], where for each n we let $p = p_n$ be a number between 0 and 1. We then make the set of all labeled graphs with n vertices into a probability space by requiring the probability of each possible edge to be p, independent of the other edges. Then we say almost all graphs have any given property P if and only if $Pr(G \in P) \to 1$ as $n \to \infty$.

Before verifying the axioms, we remark how they generalize the axioms in Section 1. The axioms of Section 1 implied the existence of "small" structures in G. The Axioms (k, l) imply G possesses "large" (i.e. of size n) structures. As indicated above, Axiom $(2, 1)$ easily implies hamiltonicity of G. Figure 1 illustrates some other structures G must contain given G satisfies certain of these axioms (in each case the structure contains all the vertices of G). In Figure 1a $F_n^{(2,1)}(S_0, <_{S_0}, \ldots, S_i, <_{S_i}, (x_1, x_2), y_1)$ adds y_1 to the top branch of the graph if $i \equiv 0 \pmod 6$, to the middle branch if $i \equiv 1, 2 \pmod 6$ and to the bottom branch if $i \equiv 3, 4, 5 \pmod 6$, each time connecting y to consecutive elements $x_1, x_2 \in S_i$. (Here we take d to be a path of length two with end vertices a_1 and a_2 and middle vertex b_1). The reader can easily discern the relevant constructing functions for the other structures. (Note in Figure 1c and 1d $k = 2$ and $l = 3$.)

Before stating the theorem, we give a few technical conditions which are required in the theorem. Let $0 < \beta < \alpha \le 1$ be fixed numbers and define $w = \min\{\frac{1}{10}(\alpha - \beta), \frac{\binom{l}{2}}{4k}(\alpha - \beta), \frac{\alpha}{10}(3\beta + \alpha), \frac{\alpha\binom{l}{2}}{4k}(3\beta + \alpha), \frac{1}{2\binom{r}{2}}\} > 0$ if $l > 1$ and $w = \min\{\frac{1}{10}(\alpha - \beta), \frac{\alpha}{10}(3\beta + \alpha), \frac{1}{2\binom{r}{2}}\} > 0$ if $l = 1$.

Next we let $0 < p_n < 1$ be a sequence of numbers and define $q_n = 1 - p_n$. Finally, let $m_n = \min\{p_n, q_n\}$.

Theorem 4.1 *Let $\{F_n^{(k,l)}\}$ be a constructing family with bounding functions f_n satisfying $f_n(m) > m^\alpha$ for all $m \ge n^\beta$, and let p_n be the probability of an edge in a labeled graph with n vertices. Suppose that there is an $\epsilon > 0$ such that if n is sufficiently large then $m_n > n^{-\frac{w}{l\binom{l}{2}} + \epsilon}$ if $l > 1$ and $m_n > n^{-w+\epsilon}$ if $l = 1$. Then almost all graphs satisfy Axiom (k, l).* □

Before giving the proof we point out the following two lemmas which will be used in the proof of Theorem 4.1. Let $\mathbb{Z}_n$ represent the integers modulo n.

Lemma 4.2 *Let $V = \mathbb{Z}_l \times \mathbb{Z}_n$. For each $0 \le i \le \frac{n}{l}$ and $0 \le j < n$, let $T_{i,j} = \{(r, j + ri) | 0 \le r < l\} \subset V$, where the second coordinate is interpreted modulo n. Then for any $(i,j) \ne (k,t)$ with $0 \le i,k \le \frac{n}{l}$ and $0 \le j,t < n$, $|T_{i,j} \cap T_{k,t}| < 2$.* □

The proof of Lemma 4.2 is straightforward. We call $\{T_{i,j} | i = i_0\}$ the i_0 *sweep*. Note that in each sweep each pair in V occurs exactly once.

Next we need a technical lemma which allows us to partition the vertex set of almost all graphs into the the basic building blocks needed in Theorem 4.1. Given a "probabilistic" construction based on a random graph, we say that an edge $\{v, w\}$ is **used** if the estimate of the probability that the construction succeeds depends on whether or not $\{v, w\}$ is an edge in the graph. Otherwise the edge is said to be **unused**. So, if an edge is unused, the success of the construction is independent of whether or not the graph has the given edge. In the proofs below we will ignore the difference between terms such as $\sqrt{x}$ and $[\sqrt{x}]$ which do not change the calculations in any significant manner.

Lemma 4.3 *Suppose p_n, the probability of any given edge in a labeled graph with n vertices, satisfies $m_n > n^{-\frac{w}{l\binom{l}{2}}+\epsilon}$ if $l > 1$ and $m_n > n^{-w+\epsilon}$ if $l = 1$ for some $\epsilon > 0$ and all n sufficiently large. Let d be a graph with l vertices and d_{in} be a graph with $r \le k + l$ vertices. Then almost all graphs can be partitioned into one set of size r which induces a subgraph isomorphic with d_{in}, $\left[\frac{n-r}{l}\right]$ of size l each inducing a subgraph isomorphic with d, and one with the residual vertices inducing a subgraph isomorphic with the induced graph on the first vertices of d. Furthermore, with this construction given any vertex v, there are at most $O(n^{\frac{5w}{2}+\epsilon})$ used edges involving v.*

Proof. Let $n \in \mathbb{N}$ and consider the probability space of all labeled graphs on n vertices. We identify the vertex set of the graphs with $V = \{1, 2, \ldots, n\}$. Let V_0 be the last $\sqrt{n}$ vertices of V. We partition the set V_0 into $\frac{\sqrt{n}}{r}$ sets of consecutive integers each of size r (with perhaps a few residual vertices). The probability that the induced subgraph on at least one of these subsets is isomorphic with d_{in} is bounded below by $1 - (1 - m_n^{\binom{r}{2}})^{\frac{\sqrt{n}}{r}}$. By taking logarithms and using the fact that $w \le \frac{1}{2\binom{r}{2}}$, it follows that this probability approaches one as n approaches infinity. Note that in this construction each vertex has at most only r used edges and if $\{i, j\}$ is used then both i and j are in V_0. Assuming there is such a subgraph we call it D_{in}.

The next step is to partition the rest of the vertices into subsets of size l so the induced graph on each subset is isomorphic with d. If $l = 1$ then this step is trivial since d consists of just a vertex. So we assume here that $l > 1$. To this end, we partition the vertices of $V_1 = V - V_0$ into two subsets, one consisting of the first $b|V_1|$ vertices of V, which we call V_2, and the other of size $(1 - b)(n - \sqrt{n})$, which we call V_3, where $b = n^{-w}$. We further partition V_3 into two subsets, V_4 with $b^2 n = lu$ vertices and V_5 with $|V_3| - |V_4|$ vertices. We follow the following algorithm until $V_5 = \emptyset$ or for $n^{1+\frac{w}{2}}$ steps, whichever comes first.

We index the vertices of V_4 by $\mathbb{Z}_l \times \mathbb{Z}_u$. We list the sets of Lemma 4.2 in the order $T_{0,0}, T_{0,1}, \ldots, T_{0,u-1}, \ldots, T_{\frac{u}{l},0}, \ldots, T_{\frac{u}{l},u-1}$. Starting with the first set $S = T_{0,0}$ we check if the subgraph induced by the vertices with indices in S is isomorphic with d (A success is when the induced subgraph is isomorphic with d and a failure is when it is not isomorphic with d). In the case of a success, we add the vertices indexed by S to the set W (which will contain sets of size l each inducing a subgraph isomorphic to d), remove the next l vertices from V_5 and index them by the indices in S. We then proceed to the next step which is to set S to the next set in the list of $T_{i,j}$ and continue as before. Note that the number of $T_{i,j}$s is $O(n^{2(1-2w)}) > n^{1+\frac{w}{2}}$ since $w < \frac{1}{10}(\alpha - \beta) < \frac{2}{9}$.

At each step the probability of a success is bounded below by $m_n^{\binom{l}{2}} \geq n^{-\frac{w}{2}+\epsilon\binom{l}{2}}$. Since at each step only unused edges are considered, the probability of exhausting V_5 before $n^{1+\frac{w}{2}}$ steps is bounded below by the probability that in $n^{1+\frac{w}{2}}$ independent Bernoulli trials with probability of success $m_n^{\binom{l}{2}}$ one attains at least $O(n)$ successes. It is clear that this probability approaches 1 as n approaches infinity using, for example Chebychev's inequality $Pr(|X-\mu| > a\sigma) < \frac{1}{a^2}$. Furthermore, by the ordering of the sets $T_{i,j}$ it is clear that for each vertex v at most $\frac{n^{1+\frac{w}{2}}}{u} = O(n^{\frac{5w}{2}})$ edges involving v are used, and each of the used edges has both vertices in V_3.

At this point in the algorithm we have one subgraph, D_{in}, which is isomorphic to d_{in}, and W which contains $\frac{(1-b)(n-\sqrt{n})-b^2n}{l}$ subgraphs each isomorphic with d. The rest of the vertices are in a subset, $V_6 \subset V_4 \cup (V_0 - D_{in})$ of size approximately b^2n and V_2 of size approximately bn. First note that no edge between V_6 and V_2 is used. We next find $|V_6|$ subsets of size l from $V_2 \cup V_6$ by partitioning the set V_2 into approximately $\frac{bn}{l-1}$ subsets of size $l-1$ each and attempting to match up each vertex in V_6 with a subset of size $l-1$ from V_2 such that the induced subgraph from these vertices is isomorphic to d. For each vertex in V_6 we examine $\frac{1}{(l-1)b}$ subsets of size $l-1$ from V_2 to check if one of the induced subgraphs is isomorphic with d. The probability that we succeed for each vertex in V_2 is bounded below by $(1-(1-m_n^{\binom{l}{2}})^{\frac{1}{b(l-1)}})^{b^2n}$. By taking logarithms it is easy to check this approaches 1 as n approaches infinity.

Next, the remaining vertices in V_2 need to be partitioned into subsets of size l so each induced subgraph is isomorphic to d. We do this by first partitioning the remaining vertices of V_2 into subsets of size l (with perhaps one of the subsets having size less than l). We then index the subsets in the partition of W with $\mathbb{Z}_{l-1} \times \mathbb{Z}_{n'}$ where $n' = O(n)$. We define $T_{i,j}$ as in Lemma 4.2 and order the $T_{i,j}$s as before. The subsets in the partition of V_2 are ordered arbitrarily.

Note that $T_{i,j}$ consists of $l-1$ subgraphs each isomorphic to d. We call an index set, $T_{i,j}$, bad if there is a used edge between two of the subgraphs indexed by $T_{i,j}$, otherwise the index set is good. We call sweep i_0 acceptable if no more that half of the index sets $T_{i_0,j}$ in the sweep are bad. Note that there are at most $O(n^{\frac{5}{2}w})$ sweeps which are not acceptable since the total number of used edges is $O(n^{1+\frac{5}{2}w})$. Therefore there are at least $O(n) > |V_2|$ acceptable sweeps.

To each subset Y in the partition of V_2 we associate an acceptable sweep. For each of the first n^w good sets S in that sweep we attempt to repartition the vertices of S and Y into l subsets of size l so that each subset induces a subgraph isomorphic with d. Specifically we arbitrarily number the vertices in each set of S from 1 to l and number the vertices in Y from 1 to l and then check if the vertices having the same number form a graph isomorphic with d.

It is easy to verify that the probability that we succeed for every Y is bounded below by $\left(1-\left(1-m_n^{l\binom{l}{2}}\right)^{n_1}\right)^{n_2}$ where $n_1 = O(n^w)$ and $n_2 = O(n^{1-w})$. It is easily checked that this probability bound approaches 1. Clearly for each vertex v at most $O(n^w)$ edges containing v become used in this process. □

Now we are ready to prove Theorem 4.1. The proof is much like the proof of Lemma 4.3, in that the same ideas of partitioning, looking for edges, and estimating probabilities of success are used.

Proof. For a fixed constructing family of functions $\{F_n^{(k,l)}\}$ with bounding functions f_n, and α and β as above, we fix d and d_{in} and consider the probability that a graph G on n vertices satisfies Axiom $A^{(k,l)}$. To begin, let G be a graph with n vertices. We assume that $n \equiv |d_{in}| \pmod{|d|}$ in order to avoid inessential detail relating to how to deal with the special case at the end where fewer than $|d|$ vertices are left over. We first use Lemma 4.3 to partition the vertex set of G into one subset S_0, inducing a subgraph of G isomorphic with d_{in}, and the rest into subsets of size l, each inducing a subgraph of G isomorphic with the subgraph of d induced by the vertices $b_1, \ldots, b_l$.

Let $s = \frac{n-|d_{in}|}{l}$ and $h_1, \ldots, h_s$ be the sets in the partition defined above. We partition the set $\{h_1, \ldots, h_s\}$ into two subsets, A and B with $|A| = s^{\frac{\alpha+\beta}{2}}$ and $|B| = s - s^{\frac{\alpha+\beta}{2}}$. We further partition A into $j = s^{\frac{3}{4}\beta+\frac{1}{4}\alpha}$ sets, each of size $s^{(\frac{1}{4}\alpha-\frac{1}{4}\beta)}$, which we call $T_1, T_2, \ldots, T_j$.

We now attempt to form $S_1, S_2, \ldots, S_j$ as in the statement of $A^{(k,l)}$. Given the sets $S_1, S_2 \ldots, S_i$, we look for $h \in T_{i+1}$ such that if $(\tilde{c}_1, \ldots, \tilde{c}_k)$ are the $<_{S_i}$ first k vertices in S_i (with order $<_{S_i}$ given inductively from $F_n^{(k,l)}$), then the subgraph of G induced from $\{\tilde{c}_1, \ldots, \tilde{c}_k\} \cup h$ fits d. Since the construction of Lemma 4.3 used at most $O(n^{\frac{5w}{2}})$ edges for each vertex, there are at least $s^{\frac{1}{4}\alpha-\frac{1}{4}\beta} - O(n^{\frac{5w}{2}})$ choices of sets in T_{i+1} which involve unused edges. It follows that the probability that the first $s^{\frac{3}{4}\beta+\frac{1}{4}\alpha}$ steps can be accomplished is at least

$$P_1 = \left(1 - \left(1 - m_n^{kl}\right)^{s^{\frac{1}{4}\alpha-\frac{1}{4}\beta} - O(n^{\frac{5w}{2}})}\right)^{s^{\frac{3}{4}\beta+\frac{1}{4}\alpha}}$$

We let S denote the vertices and $<_S$ denote the order produced to this point.

Next we let the sets in B be denoted $U_1, U_2, \ldots, U_{s-s^{\frac{\alpha+\beta}{2}}}$ and succesively adjoin the l vertices $U_{i+1} = \{\tilde{b}_1, \ldots, \tilde{b}_l\}$ to k consecutive vertices $c_1, \ldots, c_k$ in $S \cup (U_1 \cup \cdots \cup U_i)$, (with order given inductively by $F_n^{l,k}$) so that the subgraph of G induced by $\{c_1, \ldots, c_k\} \cup U_{i+1}$ fits d. An easy computation gives the following lower bound for the probability of success for every U_i as

$$P_2 = \left(1 - \left(1 - m_n^{kl}\right)^{\frac{\left(l s^{(\frac{3}{4}\beta+\frac{1}{4}\alpha)}\right)^{\alpha}}{k} - lO(n^{\frac{5w}{2}})}\right)^{s-s^{\frac{\alpha+\beta}{2}}}.$$

Last, we denote the subsets of size l in $A-S$ by V_1, V_2, ..., and at step $i+1$ attempt to join V_{i+1} to k consecutive vertices in $S \cup B \cup (V_1 \cup \cdots \cup V_i)$ as required. Now at each step there are at least $(l(s-s^{\frac{\alpha+\beta}{2}}))^{\alpha} - ls^{\frac{\alpha+\beta}{2}}$ vertices in the domain of the linear ordering given by $F_n^{(k,l)}$ which lie in B. Since there are only $ls^{\frac{\alpha+\beta}{2}}$ vertices in A, at least

$$\frac{(l(s-s^{\frac{\alpha+\beta}{2}}))^{\alpha} - ls^{\frac{\alpha+\beta}{2}}}{ls^{\frac{\alpha+\beta}{2}}} \geq \frac{1}{2} s^{\frac{\alpha-\beta}{2}} l^{\alpha-1}$$

(for large enough n) vertices in B must be consecutive in that ordering. This easily gives the following lower bound for the probability of success:

$$P_3 = \left(1 - \left(1 - m_n^{kl}\right)^{\frac{l^{\alpha-1}}{2k} s^{\frac{\alpha-\beta}{2}} - lO(n^{\frac{5w}{2}})}\right)^{s^{\frac{\alpha+\beta}{2}}}$$

Since only unused edges were considered in the estimates of probabilities P_1, P_2, and P_3, we get $P_1P_2P_3$ as a lower bound for the probability that the graph satisfies $A^{(k,l)}$ given that Lemma 4.3 is satisfied for fixed d and d_{in}. Let P_4 be the minimum probability that Lemma 4.3 is satisfied for a graph with n vertices over all d and d_{in} with parameters r, l, k. As we range over all possible choices for d and d_{in} we get $(1 - 2^{\binom{r}{2}} 2^{kl} 2^{\binom{l}{2}} (1 - P_1P_2P_3P_4))$ as a lower bound for the probability that G satisfies axiom $A^{(k,l)}$ for the given $F_n^{(k,l)}$. It is enough to show $P_i \to 1$ as $n \to \infty$ for $i = 1, 2, 3$. This follows easily in all three cases by taking logarithms. We point out that the bounds $w < \frac{\alpha-\beta}{10}$ and $w < \frac{\alpha-\beta}{4k}\binom{l}{2}$ are needed for estimating P_1, $w < \frac{\alpha}{10}(3\beta+\alpha)$ and $w < \frac{\alpha\binom{l}{2}}{4k}(3\beta+\alpha)$ are needed for estimating P_2 and $w < \frac{\alpha-\beta}{5}$ (which follows from $w < \frac{1}{10}(\alpha-\beta)$) and $w < \frac{\binom{l}{2}}{4k}(\alpha-\beta)$ are needed for estimating P_3. □

We consider now a variation $\tilde{A}^{(k,l)}$ of the axiom scheme $A^{(k,l)}$. Here we do not allow negative edge requirements.

Axiom $\tilde{\mathbf{A}}^{(\mathbf{k},\mathbf{l})}$: Let d be a graph with vertex set $\{a_1, \ldots, a_k, b_1, \ldots, b_l\}$ and let d_{in} be a graph with vertex set $\{c_1, \ldots, c_r\}$ where $k \leq r \leq k+l$. Then there is a sequence $(S_0, <_{S_0}), \ldots, (S_m, <_{S_m})$ with $m = \left[\frac{n-r}{l}\right]$, where S_i is a subset of the vertices of G and

1) The subgraph of G induced from the vertices S_0 is isomorphic with d'_{in} where d'_{in} has the same vertex set as d_{in} and contains all the edges of d_{in}, but may contain more edges. Also, the ordering $<_{S_0}$ on S_0 corresponds to the ordering $c_1 < c_2 < \cdots < c_r$ under the isomorphism.
2) $F_n^{(k,l)}(S_0, <_{S_0}, \ldots, S_j, <_{S_j}, (x_1, \ldots, x_k), (y_1, \ldots, y_l)) = (S_{j+1}, <_{S_{j+1}})$ for some elements $x_1, \ldots, x_k \in S_j$, $y_1, \ldots, y_l \notin S_j$, and where $x_1, \ldots, x_k$ are consecutive elements of S with respect to $<_{S_j}$. Also, x_iAy_s in G if a_iAb_s and y_iAy_j in G if b_iAb_j in d.
3) If $e = n - \left(r + l\left[\frac{n-r}{l}\right]\right) > 0$, then there are $y_1, \ldots, y_e$, vertices in G which are not in S_m, and consecutive elements (using order $<_{S_m}$) $x_1, \ldots, x_k$ such that x_iAy_s in G if a_iAb_s and y_iAy_j in G if b_iAb_j in d.

We have the analog of Theorem 4.1.

Theorem 4.4 *Let $\{F_n^{(k,l)}\}$ be a constructing family with bounding function f_n for $F_n^{(k,l)}$. Suppose that for all $c > 0$, for all sufficiently large n, $f_n(m) > c \log m$ for all $n^\gamma < m \le n$ for some $\gamma < 1$. If p_n, the probability of any given edge on a labeled graph with n vertices is bounded below by $\frac{L}{(\log n)^{\frac{1}{kl}}}$ for some constant L, then the probability that a graph on n vertices satisfies $\tilde{A}^{(k,l)}$ tends to 1 as $n \to \infty$.*

Proof. We fix a suitable large $c > 0$ and only consider n such that $f_n(m) > c \log m$ for $n^\gamma < m \le n$. Let $\gamma' = \frac{1+\gamma}{2}$. We exploit a device of [4]. Instead of considering graphs with probability of an edge p_n, we consider three graphs with the same vertex set and color the edges in one graph red, one graph blue, and the other graph green. We denote the probability of a red edge by p_r, the probability of a blue edge by p_b and the probability of a green edge p_g. We then define the graph G to have the same vertex set as the three graphs and let $\{v, w\}$ be an edge of G if and only if $\{v, w\}$ is either a blue, red or green edge. Note that the probability that $\{v, w\}$ is an edge in G is $1 - (1 - p_r)(1 - p_b)(1 - p_g)$ and the edges are independent. By choosing $p_r = p_b = p_g = 1 - \sqrt[3]{1 - p_n}$ we get p_n for the probability of each edge in G. (Note that for small p_n, the common value of $p_r = p_b = p_g$ is approximately $\frac{1}{3}p_n$.)

We first apply Lemma 4.3 using the green graph. (Note that the probability bound in Lemma 4.3 is satisfied.) Let P_0 denote the probability that the green graph satisfies Lemma 4.3. We will call each of the subgraphs isomorphic to d a block.

We next partition the first $n^{\gamma'}$ blocks of G into n^γ disjoint sets, each of size $n^{\gamma'-\gamma}$, say $T_0, T_1, \ldots$. We start with d_{in} and successively find a block in T_{i+1} that can be added to $S_0 \cup S_1 \cup \cdots \cup S_i$ according to the function $F^{(k,l)}$ using only blue edges. A computation as before yields the following lower bound for the probability of success:

$$P_1 = \left(1 - \left(1 - p_b^{kl}\right)^{n^{\gamma'-\gamma}}\right)^{n^\gamma}.$$

So far approximately n^γ blocks have been included.

Next, we attempt to succesively add the remaining blocks to consecutive vertices already included using only red edges. This gives the following lower bound for the probability of success:

$$P_2 = \left(1 - \left(1 - p_r^{kl}\right)^{\frac{c \log(n^\gamma)}{k}}\right)^{\frac{n}{l}}.$$

It is easy to check that P_0, P_1 and P_2 all approach one as n approaches infinity. □

We leave it to the reader to check that if we replace $c \log n$ in the statement of Theorem 4.4 with the bound for $f_n(m)$ in Theorem 4.1 then one can improve the estimate for p_n in Theorem 4.4 to the requirement for p_n in Theorem 4.1.

Suppose we fix $0 < p < 1$ and let $p_n = p$. It is interesting to note that Theorem 4.1 implies that almost all graphs contain the structure a) in Figure 2 (using $\alpha = \frac{1}{3}$, $\beta = \frac{3}{16}$ and a suitable $\{F_n^{(k,l)}\}$ as described earlier), while Theorem 4.4 implies almost all graphs contain the structure b) (which has only positive requirements). Neither theorem implies almost all graphs contain the graph of c). We do not know if this is the case.

The probability estimates are by no means best possible for any given spanning subgraph. As an example, Hamiltonian cycles exist with probability approaching one in random graphs with $p_n \geq \frac{\log n + \log\log n + \omega(n)}{n}$ where $\omega(n) \to \infty$ as $n \to \infty$ (see for example [4]). The bound from Theorem 4.1 is $\frac{1}{n^{1/10+\epsilon}}$ for any $\epsilon > 0$, which is much larger than $\frac{\log n + \log\log n + \omega(n)}{n}$. In fact, Theorem 4.1 only asserts the existence of spanning subgraphs when the probability bound is n^w for some $-1 < w < 0$. However, it is easy to check for $p_n < \frac{1}{n^{2/3+\epsilon}}$ the probability approaches zero that one can partition a random graph on n vertices into $\frac{n}{3}$ 3-cycles. Of course the existence of such a partition follows from Theorem 4.1 with $p_n > \frac{1}{n^{1/90}}$. Although there is a big gap between $\frac{2}{3}$ and $\frac{1}{90}$, this example does show that a general theorem asserting the existence of both a Hamiltonian cycle and a partition of the vertices into triangles cannot do better than n^w for some $-1 < w < 0$.

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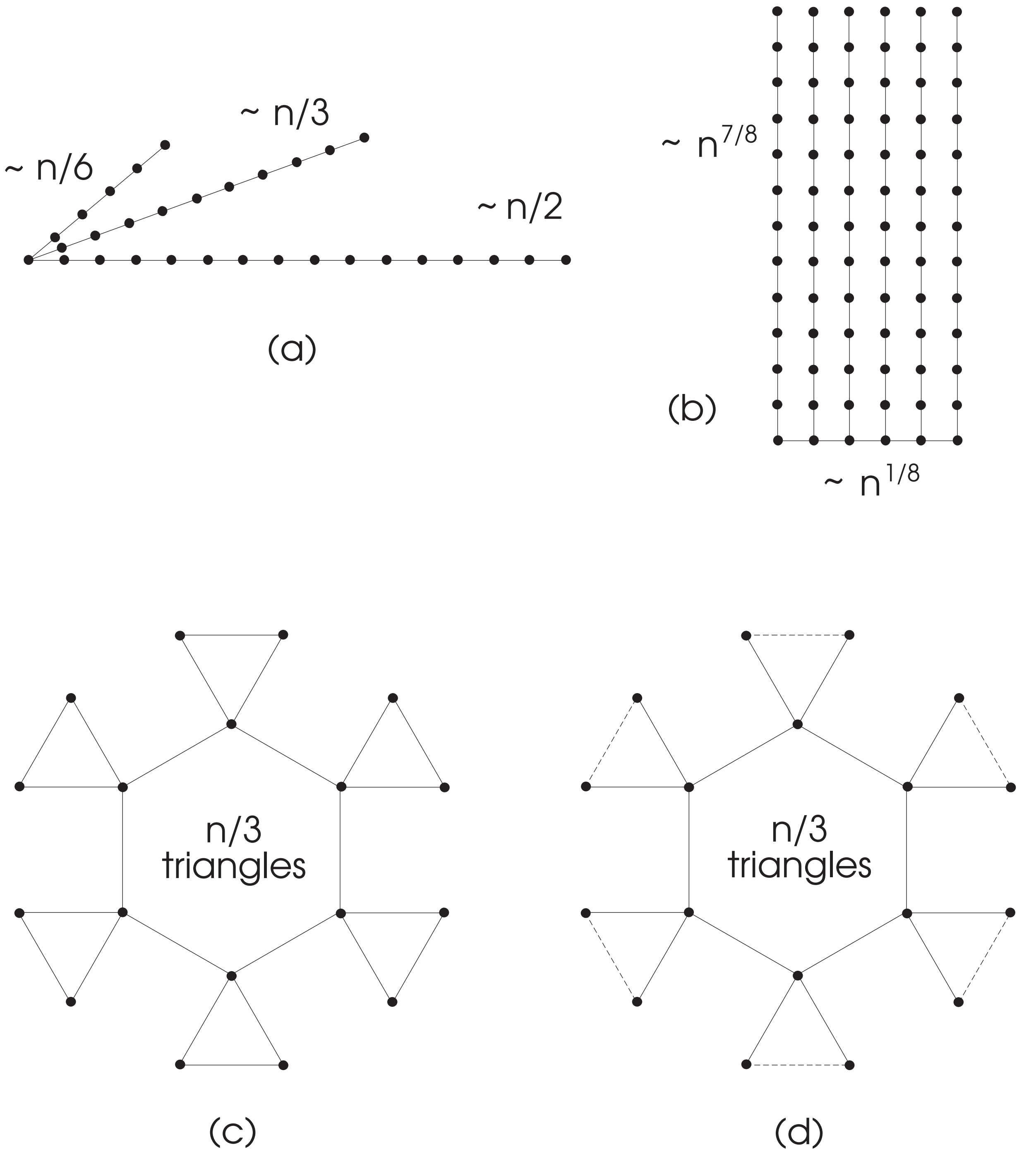

Figure 1.

Some structures of size n almost all graphs of size n contain according to Theorem 4.1. A solid line indicates an edge, a broken line indicates no edge, and neither indicates there is no requirement either for an edge or for no edge.

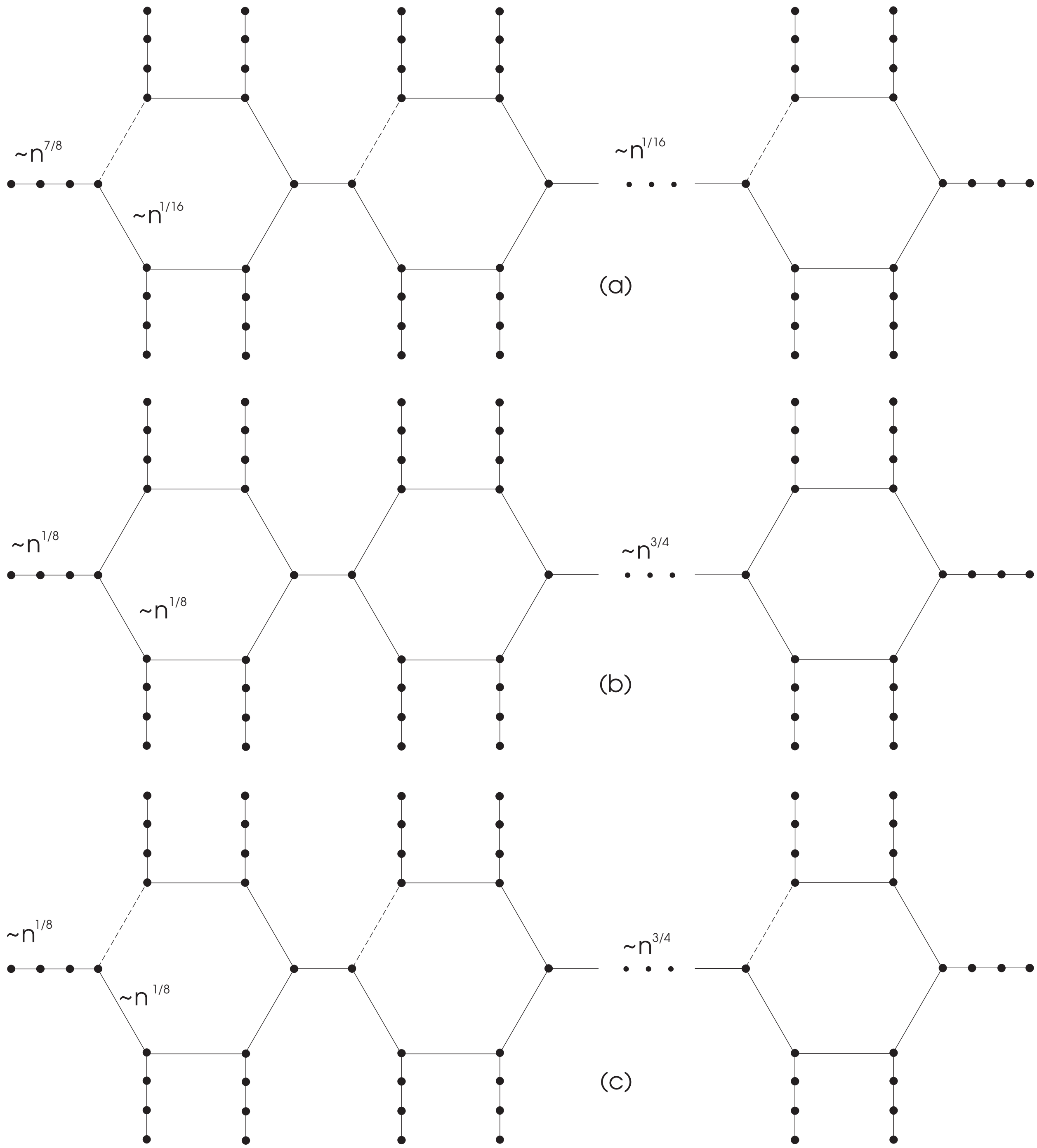

Figure 2.

As in Figure 1, a solid line indicates an edge, a broken line indicates there is no edge, and no line indicates there is no requirement concerning an edge. The graph in a) follows from Theorem 4.1, the graph in b) follows from Theorem 4.4, while the graph in c) does not follow from either theorem.