Int. Conference on Boundary and Interior Layers
BAIL 2006
G. Lube, G. Rapin (Eds)

A singular perturbation problem arising in the modelling of plasma sheaths *

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1. Introduction

Consider the interaction of a flowing plasma with a planar Langmuir probe [2]. Assume that the plasma flows in the plane of the probe surface and that the plasma consists solely of positive ions with density n_+ and electrons with density n_e. Downstream of the probe, the ions are moving with a velocity of $\mathbf{u} = (u_i, u_v + u_F)$ and (u_0, u_F) is the constant flow velocity of the ions upstream of the probe. We wish to consider the influence of the probe on the flow of the ions. Let X be the horizontal distance to the right of the probe and Y the distance along the probe (from the tip of the probe). Assuming a collision-less plasma and that the ions are cold, the continuity equations for the ion density and momentum are [2]

$$\begin{aligned} \frac{\partial n_+}{\partial t} + \nabla \cdot (n_+ \mathbf{u}) &= 0, \\ m_+ n_+ \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) &= e n_+ \mathbf{E}, \end{aligned}$$

where $\mathbf{E} = (E_x, E_y) = -\nabla \tilde{\phi}$ is the electric field and m_+ is the mass of the ions. Our interest is in the steady state case and if $u_F >> u_v$, we disregard terms involving u_v. Hence this system is approximated with the system

$$\begin{aligned} \frac{\partial}{\partial X}(n_+ u_i) + u_F \frac{\partial n_+}{\partial Y} &= 0, \\ u_i \frac{\partial u_i}{\partial X} + u_F \frac{\partial u_i}{\partial Y} &= \frac{e}{m_+} E_X, \end{aligned}$$

where the electric field is determined from solving Poisson's equation (assuming $E_X >> E_Y$)

$$-\frac{\partial E_X}{\partial X} = \frac{\partial^2 \tilde{\phi}}{\partial X^2} = \frac{e(n_+ - n_e)}{\epsilon_0}.$$

Since $m_e << m_+$ and we assume that the electrostatic potential $\tilde{\phi}$ tends to zero as one moves away from the probe, the electron density n_e is approximately related to the electrostatic potential $\tilde{\phi}$ as

$$n_e = n_0 exp(-\frac{e\tilde{\phi}}{kT_e})$$

where T_e is the electron temperature, k is Boltzmann's constant, ϵ_0 is the permittivity of free space and e is the electron charge.

In a similar fashion to the scaling of the variables used in [3], we introduce the non-dimensional independent variables x, y and the non-dimensional dependent variables n, u and ϕ, which are defined as follows:

$$n = \frac{n_+}{n_0}, \quad u = \frac{u_i}{c_s}, \quad \phi = \frac{e\tilde{\phi}}{kT_e}, \quad x = \frac{X}{L} \quad y = \frac{Y c_s}{u_F L}$$

*This research was supported in part by the National Center for Plasma Science and Technology Ireland

where the ion sound speed c_s and the electron Debyre length λ_D are defined by

$$c_s^2 = \frac{kT_e}{m_+}, \quad \lambda_D^2 = \frac{\epsilon_0 kT_e}{n_0 e^2}.$$

The length L is a distance sufficiently far from the probe so that the effect of the probe on the plasma at this distance is negligible. After reformulating the problem with the above transformations and formulating suitable boundary and initial conditions we propose to examine the related mathematical problem : Find $(u(x,y), n(x,y), \phi(x,y))$ which satisfy the following system of differential equations in the space domain $(x,y) \in (0,1) \times (0,T]$

$$\frac{\partial n}{\partial y} + \frac{\partial(nu)}{\partial x} = 0, \quad (x,y) \in [0,1) \times (0,T], \tag{1a}$$

$$\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial x}, \quad (x,y) \in [0,1) \times (0,T], \tag{1b}$$

$$\varepsilon^2 \frac{\partial^2 \phi}{\partial x^2} = e^{\phi} - n, \quad (x,y) \in (0,1) \times (0,T], \qquad \varepsilon = \lambda_D L^{-1}, \tag{1c}$$

subject to the following set of boundary and initial conditions

$$\phi(0,y) = -A,\ \phi(1,y) = 0,\ y \geq 0; \quad \phi(x,0) = \phi_0(x),\ 0 \leq x \leq 1, \tag{1d}$$

$$n(x,0) = 1,\ 0 \leq x \leq 1; \quad n_y(1,y) = -(nu_x)(1,y),\ y \geq 0, \tag{1e}$$

$$u(x,0) = \tilde{u}_0,\ 0 \leq x \leq 1; \quad u_y(1,y) = -\phi_x(1,y),\ y \geq 0. \tag{1f}$$

The parameters $\tilde{u}_0$ and A are assumed to be known and the initial condition $\phi_0(x)$ is chosen so that $\varepsilon^2 \phi_0''(x) = e^{\phi_0(x)} - 1,\ \phi_0(0) = -A,\ \phi_0(1) = 0$. Due to the presence of the singular perturbation parameter ε layers or sheaths can appear in the solutions.

2. Shishkin mesh

Consider the following linear singularly perturbed ordinary differential equation

$$L_\varepsilon u_\varepsilon(x) \equiv -\varepsilon^2 u_\varepsilon'' + b(x) u_\varepsilon = f(x),\ x \in \Omega = (0,1) \tag{2a}$$

$$b(x) \geq \beta^2 > 0, \quad u(0) = A, \quad u(1) = B. \tag{2b}$$

The solution of problem (2) can be written as the sum $u_\varepsilon(x) = v(x) + (u_\varepsilon(0) - v(0))w_l(x) + (u_\varepsilon(1) - v(1))w_r(x)$ of a regular component $v(x)$ and two singular components $w_l(x),\ w_r(x)$. The following parameter explicit bounds on these components and their derivatives can be established [1]for $0 \leq k \leq 4$

$$\|\frac{d^k v}{dx^k}\|_{\Omega,\infty} \leq C(1 + \varepsilon^{2-k}), \quad |\frac{d^k w_l}{dx^k}(x)| \leq C\varepsilon^{-k} e^{-\beta x/\varepsilon}, \quad |\frac{d^k w_r}{dx^k}(x)| \leq C\varepsilon^{-k} e^{-\beta(1-x)/\varepsilon}$$

where C is a constant that is independent of ε. Note that the singular component w_l is negligible away for $x \geq \tau = \frac{2\varepsilon \ln 1/\varepsilon}{\beta}$. However, within the region $(0, C\varepsilon)$ the derivatives of the singular component become unbounded as $\varepsilon \to 0$. To obtain a reasonable numerical approximation to the solution u_ε using N mesh intervals (where in general $\varepsilon << N^{-1}$) it is necessary [1] to use a non-uniform mesh so that a significant proportion of the mesh elements are within the layer regions. One way to achieve this is to use a piecewise uniform Shishkin mesh [1]. For the boundary value problem (2) with the additional assumption that

$$b(1) u_\varepsilon(1) = f(1) \tag{2c}$$

an appropriate Shishkin mesh is defined as follows. The domain $\overline{\Omega} = [0, 1]$ is subdivided into the two subintervals $[0, \sigma]$ and $[\sigma, 1]$. On each subinterval a uniform mesh with $\frac{N}{2}$ mesh-intervals is placed. The interior points of the mesh are denoted by $\Omega_\varepsilon^N = \{x_i : 1 \leq i < N\}$ and

$$\sigma = \min\{\frac{1}{2},\ 2\frac{\varepsilon}{\beta}\ln N\}. \tag{3}$$

The fine mesh and the coarse mesh step are given by h and H, respectively. The fitted mesh method for problem (2) is: Find a mesh function U_ε^N such that

$$-\varepsilon^2\delta^2 U_\varepsilon^N(x_i) + b(x_i)U_\varepsilon^N(x_i) = f(x_i) \quad \text{for all} \quad x_i \in \Omega_\varepsilon^N \tag{4a}$$

$$U_\varepsilon^N(0) = u_\varepsilon(0), \quad U_\varepsilon^N(1) = u_\varepsilon(1) \tag{4b}$$

where δ^2 is the standard centered finite difference operator defined for any mesh function Z by

$$\delta^2 Z(x_i) = (\frac{Z(x_{i+1}) - Z(x_i)}{x_{i+1} - x_i} - \frac{Z(x_i) - Z(x_{i-1})}{x_i - x_{i-1}})\frac{1}{x_{i+1} - x_{i-1}}.$$

Let $\bar{U}_\varepsilon^N$ denote the piecewise linear interpolant of the numerical solution U_ε^N from the mesh Ω_ε^N to the domain $[0, 1]$. The following parameter-uniform pointwise error bound follows from the results in [1]

$$\|u_\varepsilon - \bar{U}_\varepsilon^N\|_{[0,1],\infty} \leq C(N^{-1}\ln N)^2 \tag{5}$$

where C is a constant independent of the singular perturbation parameter ε. Moreover, parameter uniform estimates on the discrete derivatives of the solution can be obtained on these piecewise-uniform Shishkin meshes. Using the crude bound

$$|D^+(U_\varepsilon^N - u_\varepsilon)| \leq \frac{1}{(x_{i+1} - x_i)}(|(U_\varepsilon^N - u_\varepsilon)(x_{i+1})| + |(U_\varepsilon^N - u_\varepsilon)(x_i)|$$

and noting that $H \geq CN^{-1}$, $\varepsilon \leq CNh$, we deduce that

$$|D^+(U_\varepsilon^N - u_\varepsilon)(x_i)| \leq \begin{cases} CN^{-1}(\ln N)^2, & \text{if } x_i \geq \sigma, \\ CN^{-1}(\ln N)^2/\varepsilon, & \text{if } x_i < \sigma. \end{cases} \tag{6}$$

On the piecewise-uniform mesh we have the following result.

Lemma 1 Let U_ε^N and u_ε be the solutions of (4) and (2), then

$$|(D^+U_\varepsilon^N - u_\varepsilon')(x_i)| \leq \begin{cases} CN^{-1}(\ln N)^2, & \text{if } x_i \geq \tau = \frac{2\varepsilon \ln 1/\varepsilon}{\beta}, \\ CN^{-1}(\ln N)^2/\varepsilon, & \text{if } x_i < \tau. \end{cases}$$

Proof: Note that if the mesh is uniform (when $\sigma = 0.5$) then a classical argument suffices.

$$|D^+u - u'| \leq CN^{-1}\|u''\| \leq CN^{-1}\varepsilon^{-2} \leq CN^{-1}(\ln N)^2.$$

Now we assume that $\sigma < 0.5$. Consider first the case of $\varepsilon \leq N^{-1}$, which implies that $\sigma \leq \tau$. Then for $x_i \geq \tau$

$$\begin{aligned} |D^+u - u'| &\leq |D^+v - v'| + |D^+w_l - w_l'| \quad \leq CN^{-1}\|v''\| + CN^{-1}\|w_l''\|_{[\tau,1)} \\ &\leq CN^{-1}, \text{ since } |w_l''(x)| \leq C\varepsilon^{-2}e^{-\beta\tau/\varepsilon} \leq C,\ x_i \geq \tau. \end{aligned}$$

For $\sigma \leq x_i < \tau$

$$\begin{aligned} \varepsilon|D^+u - u'| &\leq \varepsilon|D^+v - v'| + \varepsilon|D^+w_l - w_l'| \ \leq CN^{-1}\varepsilon\|v''\| + C\varepsilon\|w_l'\|_{[\sigma,1)} \\ &\leq CN^{-2}, \quad \text{since} \quad \varepsilon|w_l'(x)| \leq Ce^{-\beta\sigma/\varepsilon} \leq CN^{-2},\ x_i \geq \sigma. \end{aligned}$$

For $x_i < \sigma$

$$\begin{aligned}\varepsilon|D^+u - u'| &\le \varepsilon|D^+v - v'| + \varepsilon|D^+w_l - w_l'| \le C\varepsilon h\|v''\| + C\varepsilon h\|w_l''\| \\ &\le CN^{-1}\ln N, \quad \text{since} \quad h \le C\varepsilon N^{-1}\ln N.\end{aligned}$$

In the other case of $\varepsilon > N^{-1}$ then $\sigma > \tau$. For $x_i \ge \tau$

$$|D^+u - u'| \le CN^{-1}\|v''\| + CN^{-1}\|w_l''\|_{[\tau,1)} \le CN^{-1}$$

and for $x_i < \tau < \sigma$

$$\varepsilon|D^+u - u'| \le CN^{-1}\|v''\| + C\varepsilon h\|w_l''\| \le CN^{-1}\ln N.$$

Combine these bounds with (6) to complete the proof.

Consider the higher order discrete approximation to the derivative defined by

$$D^0Z(x_i) = \frac{h_iD^+Z(x_i) + h_{i+1}D^-Z(x_i)}{h_{i+1}+h_i}, \quad h_i = x_i - x_{i-1}.$$

The solution of problem (2) can also be written as the sum of a modified regular component $\tilde{v}(x)$ and the two singular components $w_l(x),\ w_r(x)$ so that the following bounds can be established

$$\|\frac{d^k\tilde{v}}{dx^k}\|_{\Omega,\infty} \le C(1+\varepsilon^{4-k}), \quad 0 \le k \le 4.$$

From these bounds it follows that at all internal mesh points

$$|D^0\tilde{v}(x_i) - \tilde{v}'(x_i)| \le CN^{-2}.$$

Consider a two-transition point mesh $\overline{\omega}_2$ where the transition points are taken to be

$$\sigma_2 = 2\min\{\frac{1}{8},\ \frac{\varepsilon}{\beta}\ln N,\ \frac{\varepsilon}{\beta}\ln\frac{1}{\varepsilon}\},\ \tau_2 = 4\min\{\frac{1}{8}, \max\{\ \frac{\varepsilon}{\beta}\ln N,\ \frac{\varepsilon}{\beta}\ln\frac{1}{\varepsilon}\}. \tag{7}$$

and a uniform mesh with $\frac{N}{4}, \frac{N}{4}, \frac{N}{2}$ mesh-intervals is used in each of the subdomains $[0,\sigma_2] \cup [\sigma_2,\tau_2] \cup [\tau_2,1]$.

Lemma 2 For the mesh points $x_i \in \omega_2$ we have that

$$|(D^0w_l - w_l')(x_i)| \le \begin{cases} CN^{-2}(\ln N)^2, & \text{if } x_i > \tau_2, \\ CN^{-2}(\ln N)^2/\varepsilon, & \text{if } x_i \le \tau_2. \end{cases}$$

Proof The case of $\tau_2 = 0.5$ is dealt with in a classical way. Assume that $\tau_2 < 0.5$. Consider first the case of $\varepsilon \le N^{-1}$, which implies that

$$\sigma_2 = 2\frac{\varepsilon}{\beta}\ln N, \quad \tau_2 = 4\frac{\varepsilon}{\beta}\ln(1/\varepsilon).$$

Then for $x_i > \tau_2$, where the mesh is uniform,

$$|D^0w_l - w_l'| \le CN^{-2}\|w_l'''\|_{[\tau_2,1)} \le CN^{-2}\varepsilon,$$

since $|w_l'''(x)| \le C\varepsilon^{-3}e^{-\beta\tau_2/\varepsilon} \le C\varepsilon,\ x_i \ge \tau_2$. For $\sigma_2 \le x_i \le \tau_2$

$$\varepsilon|D^0w_l - w_l'| \le C\varepsilon\|w_l'\|_{[\sigma-h_1,1)} \le CN^{-2}, \quad h_1 = x_1 - x_0,$$

since $\varepsilon|w_l'(x)| \le Ce^{-\beta\sigma/\varepsilon} \le CN^{-2},\ x_i \ge \sigma$ and $e^{\beta h_1/\varepsilon} \le C$. For $x_i < \sigma_2$, where the mesh is fine and uniform,

$$\varepsilon|D^0w_l - w_l'| \le C\varepsilon h_1^2\|w_l'''\|_{[0,\sigma_2]} \le C(N^{-1}\ln N)^2,$$

since $h_1 \le C\varepsilon N^{-1}\ln N$. In the second case of $\varepsilon > N^{-1}$ then

$$\sigma_2 = 2\frac{\varepsilon}{\beta}\ln\frac{1}{\varepsilon}, \quad \tau_2 = 4\frac{\varepsilon}{\beta}\ln N.$$

In the uniform mesh regions we have the following: For $x_i < \sigma_2$

$$\varepsilon|D^0 w_l - w_l'| \le C\varepsilon h_1^2\|w_l'''\| \le CN^{-2}(\ln\frac{1}{\varepsilon})^2 \le C(N^{-1}\ln N)^2,$$

for $x_i > \tau_2$

$$|D^0 w_l - w_l'| \le CN^{-2}\|w_l'''\|_{[\tau_2,1)} \le CN^{-2}\varepsilon^{-3}N^{-4} \le CN^{-3}$$

and for $\sigma_2 < x_i < \tau_2$, with $Nh_2 = 4(\tau_2 - \sigma_2)$ we have

$$\varepsilon|D^0 w_l - w_l'| \le Ch_2^2\varepsilon\|w_l'''\|_{[\sigma_2,1)} \le C(N^{-1}\ln N)^2.$$

At the second transition point $x_i = \tau_2$

$$\varepsilon|D^0 w_l - w_l'| \le C\varepsilon\|w_l'\|_{[\tau_2-h_2,1)} \le CN^{-2}$$

and, finally, for $x_i = \sigma_2$ we have that

$$\varepsilon|D^0 w_l - w_l'| \le C\varepsilon^3(N^{-1}\ln N)^2\|w_l'''\|_{[\sigma_2-h_1,1)} \le C(N^{-1}\ln N)^2.$$

3. Nonlinear ordinary differential equation

Let us now consider the nonlinear ordinary differential equation

$$-\varepsilon^2 y'' + e^y = n(x),\ x \in \Omega = (0,1), \tag{8a}$$
$$y(0) = -A,\ y(1) = 0, \quad 0 < n(x) \le 1, \qquad n(1) = 1. \tag{8b}$$

Note that the Bernoulli function

$$b(y) = \frac{e^y - 1}{y}, y \ne 0; \qquad b(0) = 1$$

satisfies

$$\frac{\partial b}{\partial y} > 0, \quad \forall y \quad \text{and} \quad 0 < \beta_1^2 = \frac{1 - e^{-A}}{A} \le b(y) \le b(0) = 1, \quad y \in [-A, 0].$$

Reformulate problem (8) into the form

$$-\varepsilon^2 y'' + b(y)y = n(x) - 1 = f(x),\ x \in \Omega = (0,1), \tag{9a}$$
$$f(1) = 0; \quad b(y) \ge \beta_1^2 > 0,\ \forall y \in [y(0), y(1)]; \quad y(0) = -A,\ y(1) = 0. \tag{9b}$$

Motivated by the linear problem (2), we propose the following nonlinear numerical method for problem (9). The domain $[0,1]$ is split into $[0,\sigma_2] \cup [\sigma_2, 1]$ and a uniform mesh is constructed on each of these subintervals. The numerical method is then: Find Y_ε^N such that

$$-\varepsilon^2\delta^2 Y_\varepsilon^N(x_i) + b(Y_\varepsilon^N)Y_\varepsilon^N(x_i) = f(x_i) \quad \text{for all} \quad x_i \in \Omega_2^N \tag{10a}$$
$$Y_\varepsilon^N(0) = y_\varepsilon(0), \quad Y_\varepsilon^N(1) = y_\varepsilon(1) \tag{10b}$$

where the transition point in the piecewise uniform mesh Ω_2^N is taken to be

$$\sigma_2 = \min\{\frac{1}{2},\ 2\frac{\varepsilon}{\beta_1}\ln N\}, \qquad \beta_1 = \sqrt{\frac{1 - e^{-A}}{A}}. \tag{11}$$

Consider the following reduced systems of two equations for a given function $n(x)$

$$\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial x}, \qquad \varepsilon^2\frac{\partial^2 \phi}{\partial x^2} = e^{\phi} - n(x).$$

Note that the forcing term in the first order equation is unbounded with respect to ε. Introduce the stretched variables $\eta = \frac{x}{\varepsilon},\ \zeta = \frac{y}{\varepsilon}$ and the transformed equation for $\hat{u}(\eta, \zeta) = u(x, y)$ is

$$\frac{\partial \hat{u}}{\partial \zeta} + \hat{u}\frac{\partial \hat{u}}{\partial \eta} = -\frac{\partial \hat{\phi}}{\partial \eta}, 0 < \eta < C, 0 < \zeta < C$$

where the forcing term is now of order one in the corner area $(\eta, \zeta) \in [0, 1] \times [0, 1]$. The above observation motivates the use of a piecewise-uniform mesh in the vertical direction.

4. Numerical Method in the case of $u_0 < 0$.

The algorithm given below involves some key elements. Firstly, due to the presence of the singular perturbation parameter in (1c), we employ a piecewise-uniform Shishkin mesh in the horizontal direction. Secondly, due to the unbounded (with respect to ε) source term in (1b) we will use a piecewise-uniform mesh in the vertical direction. Thirdly, we discretize the source term in (1b) using a discrete difference operator with second order truncation error. Finally, the step-size in time is reduced adaptively at each time level in the algorithm, if the nonlinear solver is not converging for a particular time step.

The domain $\overline{\Omega} = \overline{\Omega}_x \times \overline{\Omega}_y$ is discretized by the tensor product mesh $\overline{\Omega}^{N,M}$ where $\overline{\Omega}^{N,M} = \overline{\Omega}_x^N \times \overline{\Omega}_y^M$. The domains $\overline{\Omega}_x, \overline{\Omega}_y$ are composed of the subdomains

$$\overline{\Omega}_x = [0, \sigma_x] \cup [\sigma_x, \tau_x] \cup [\tau_x, 1], \quad \overline{\Omega}_y = [0, \sigma_y] \cup [\sigma_y, \tau_y] \cup [\tau_y, 1]$$

On each subdomain in the horizontal direction, a uniform mesh with $\frac{N}{4}, \frac{N}{4}, \frac{N}{2}$ mesh-intervals is used in the respective subdomains. The transition points in the horizontal and vertical direction are taken to be

$$\sigma_x = \sigma_y = \min\{\frac{1}{4},\ 4\frac{\varepsilon}{\beta_1}\ln N,\ 4\frac{\varepsilon}{\beta_1}\ln(1/\varepsilon)\},\ \tau_x = \tau_y = \min\{\frac{1}{2}, 2\max\{\ 4\frac{\varepsilon}{\beta_1}\ln N,\ 4\frac{\varepsilon}{\beta_1}\ln(1/\varepsilon)\}\}. \tag{12a}$$

Initially the vertical mesh step $k_j = y_j - y_{j-1}$ is set at

$$k_j = 4\sigma_y N^{-1},\ j \leq N/4,\ k_j = 4(\tau_y - \sigma_y)N^{-1},\ N/4 < j \leq N/2,\ k_j = 2(1-\tau_y)N^{-1},\ j > N/2.$$

The system of differential equations (1a,b,c) is discretized using a standard upwind finite difference operator on this piecewise uniform mesh. When solving the nonlinear difference scheme, the vertical mesh step will sometimes be reduced in size. The resulting nonlinear finite difference method is linearized using the iterative algorithm given below.

Set the initial approximation for the density to be constant throughout the domain

$$\mathcal{N}^0(x, y) = 1, \quad (x, y) \in [0, 1] \times [0, T]. \tag{13a}$$

and determine an approximation $\Phi(x_i, 0)$ to the initial potential $\phi(x_i, 0)$ using

$$\varepsilon^2\delta_x^2\Phi_j(x_i) - (\frac{e^{\Phi_{j-1}(x_i)} - 1}{\Phi_{j-1}(x_i)})\Phi_j(x_i) = 0, \quad 0 < x_i < 1,$$
$$\Phi_0(x_i) = -A + Ax_i,\ 0 \leq x_i \leq 1; \quad \Phi_j(0) = -A, \quad \Phi_j(1) = 0.$$

Hence our initial conditions are $\Phi(x_i,0)=\Phi_N(x_i),\ U(x_i,0)=u_0<0,\ \mathcal{N}(x_i,0)=1,\ 0\le x_i\le 1$. At each subsequent vertical mesh level $y=y_j>0$, an approximation $(\Phi,U,\mathcal{N})$ is generated from a sequence of approximations $(\Phi^k,U^k,\mathcal{N}^k)$, for $k=1,2,3,..K$ which are generated from

$$\varepsilon^2\delta_x^2\Phi^k(x_i,y_j)-\big(\frac{e^{\Phi^k(x_i,y_{j-1})}-1}{\Phi(x_i,y_{j-1})}\big)\Phi^k(x_i,y_j)=1-\mathcal{N}^{k-1}(x_i,y_j), \tag{13b}$$

$$D_y^-U^k(x_i,y_j)+U(x_i,y_{j-1})D_x^+U^k(x_i,y_j)=-D_x^0\Phi^k(x_i,y_j) \tag{13c}$$

$$D_y^-\mathcal{N}^k(x_i,y_j)+D_x^+(\mathcal{N}^kU^k)(x_i,y_j)=0 \tag{13d}$$

and the following set of boundary conditions at $(0,y_j),(1,y_j)$ and initial conditions at (x_i,y_{j-1})

$$\Phi^k(0,y_j)=-A,\quad \Phi^k(1,y_j)=0,\quad \Phi^k(x_i,y_{j-1})=\Phi(x_i,y_{j-1}),\quad 0\le x_i\le 1,$$
$$U^k(x_i,y_{j-1})=U(x_i,y_{j-1}),\ 0\le x_i\le 1,\qquad D_y^-U^k(1,y_j)=-D_x^-\Phi^k(1,y_j),$$
$$\mathcal{N}^k(x_i,y_{j-1})=\mathcal{N}(x_i,y_{j-1}),\ 0\le x_i\le 1,\qquad D_y^-\mathcal{N}^k(1,y_j)+(\mathcal{N}^kD_x^-U^k)(1,y_j)=0.$$

If at any level y_j

$$\min\{\frac{\|\Phi^k-\Phi^{k-1}\|_{\overline{\Omega}^N,\,\infty}}{\|\Phi^{k-1}\|_{\overline{\Omega}^N,\,\infty}},\frac{\|U^k-U^{k-1}\|_{\overline{\Omega}^N,\,\infty}}{\|U^{k-1}\|_{\overline{\Omega}^N,\,\infty}},\frac{\|\mathcal{N}^k-\mathcal{N}^{k-1}\|_{\overline{\Omega}^N,\,\infty}}{\|\mathcal{N}^{k-1}\|_{\overline{\Omega}^N,\,\infty}}\}>1$$

then the current time step k_j is halved and and a new value for $(\Phi(x_i,y_j),U(x_i,y_j),\mathcal{N}(x_i,y_j))$ is computed from $(\Phi(x_i,y_{j-1}),U(x_i,y_{j-1}),\mathcal{N}(x_i,y_{j-1}))$. The algorithm continues to iterate at the vertical level y_j until

$$\max\{\|\Phi^K-\Phi^{K-1}\|_{\overline{\Omega}^N,\,\infty},\|U^K-U^{K-1}\|_{\overline{\Omega}^N,\,\infty},\|\mathcal{N}^K-\mathcal{N}^{K-1}\|_{\overline{\Omega}^N,\,\infty}\}\le 10^{-8}.$$

When this condition is met, we define

$$\Phi(x_i,y_j)=\Phi^K(x_i,y_j),\ U(x_i,y_j)=U^K(x_i,y_j),\ \mathcal{N}(x_i,y_j)=\mathcal{N}^K(x_i,y_j),\ 0\le x_i\le 1. \tag{13e}$$

This iterative process is continued until $y_j=1$ is reached. Our primary interest is in approximations to the ion current density $j_+=-en_+u_i$. Thus in the following tables we examine the convergence behaviour of the numerical approximations

$$J^N(x_i,y_j)=(\mathcal{N}U)(x_i,y_j).$$

Approximations to the parameter–uniform order of convergence are computed using

$$p^*=\min_N\log_2\frac{D^N}{D^{2N}},\quad \text{where}\quad D^N=\max_\varepsilon||J^N-\overline{J}^{2N}||_{\Omega^N,\infty}$$

and $\overline{J}^N$ is the piecewise linear interpolant on Ω^N. For sufficiently large N, the first two tables suggest that the method is parameter-uniform for the variable J^N. From the final table, it appears that, in the case of supersonic flow the required number of time-steps is independent of ε; but, in the case of subsonic flow, the number of time-steps increase as $\varepsilon\to 0$.

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ε	Number of Intervals N					
	8	16	32	64	128	256
2^{-4}	0.68(1.74)	0.71(0.94)	0.50(0.49)	0.67(0.81)	0.83(0.92)	0.92(0.96)
2^{-5}	0.69(1.02)	0.72(0.56)	0.84(0.33)	0.80(0.782)	0.82(0.90)	0.88(0.95)
2^{-6}	0.74(0.91)	0.79(0.89)	0.83(0.69)	0.88(0.76)	0.86(0.78)	0.89(0.80)
2^{-7}	-0.35(1.06)	0.69(0.96)	0.82(0.89)	0.92(0.83)	0.94(0.80)	0.89(0.79)
2^{-8}	-0.10(-0.18)	0.16(1.00)	0.60(1.00)	0.43(1.01)	0.86(0.88)	0.98(0.81)
2^{-9}	-0.12(-0.27)	0.18(0.09)	0.60(0.86)	0.58(128)	0.77(1.06)	0.64(0.90)
2^{-10}	-0.02(-0.17)	0.19(0.12)	0.60(1.04)	0.58(0.55)	0.77(0.86)	0.82(0.86)
2^{-11}	0.07(-0.10)	0.19(0.14)	0.60(1.04)	0.58(0.55)	0.77(0.86)	0.82(0.86)
2^{-12}	0.14(-0.04)	0.18(0.14)	0.60(1.04)	0.58(0.55)	0.77(0.86)	0.82(0.86)
p^*	-0.06(-0.07)	0.54(0.40)	0.60(0.86)	0.43(0.71)	0.86(0.89)	0.70(0.86)

Table 1: Computed orders of convergence for J^N when A =10 ($A = 50$), u_0 =-2, $n_0 = 1$.

ε	Number of Intervals N					
	8	16	32	64	128	256
2^{-4}	1.01(1.00)	1.01(1.35)	0.58(0.65)	0.63(0.95)	0.83(0.99)	0.93(1.00)
2^{-5}	0.96(1.00)	1.03(1.40)	1.00(0.39)	0.94(0.62)	0.77(0.85)	0.87(0.94)
2^{-6}	0.97(1.07)	1.03(1.11)	1.09(0.82)	1.12(0.78)	0.96(0.78)	0.95(0.91)
2^{-7}	-0.25(1.30)	0.87(1.11)	1.07(1.17)	1.19(1.05)	1.18(0.95)	0.99(0.89)
2^{-8}	0.01(-0.18)	0.19(1.21)	0.82(1.20)	0.57(1.30)	1.13(1.09)	1.24(0.93)
2^{-9}	-0.08(-0.26)	0.25(0.11)	0.82(1.09)	0.72(0.86)	0.98(1.37)	0.92(1.11)
2^{-10}	-0.01(-0.15)	0.29(0.15)	0.82(1.28)	0.72(0.63)	0.98(1.10)	1.08(1.12)
2^{-11}	0.09(-0.08)	0.31(0.18)	0.82(1.28)	0.72(0.63)	0.98(1.10)	1.08(1.12)
2^{-12}	0.17(-0.02)	0.32(0.20)	0.82(1.28)	0.72(0.63)	0.98(1.10)	1.08(1.12)
p^*	-0.01(-0.07)	0.60(0.42)	0.82(1.09)	0.57(0.82)	1.12(1.10)	0.92(1.12)

Table 2: Computed orders of convergence for J^N when A =10 ($A = 50$), u_0 =-0.5, $n_0 = 1$.

ε	Number of Intervals N					
	16	32	64	128	256	512
2^{-4}	16(16)	32(32)	64(64)	128(128)	256(256)	512(512)
2^{-5}	16(16)	32(32)	64(64)	128(128)	256(256)	512(512)
2^{-6}	16(24)	32(32)	64(64)	128(128)	256(256)	512(512)
2^{-7}	16(24)	32(48)	64(64)	128(128)	256(256)	512(512)
2^{-8}	16(24)	32(48)	64(96)	128(128)	256(256)	512(512)
2^{-9}	16(24)	32(48)	64(96)	128(192)	256(256)	512(512)
2^{-10}	16(24)	32(80)	64(160)	128(320)	256(384)	512(768)
2^{-11}	16(24)	32(144)	64(288)	128(576)	256(640)	512(1280)
2^{-12}	16(40)	32(272)	64(544)	128(1088)	256(1152)	512(2304)

Table 3: Iteration counts for $A = 50$, u_0 =-2 (u_0 =-0.5), $n_0 = 1$.