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Characterising Concurrent Histories

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Abstract. Non-interleaving semantics of concurrent systems is often expressed using posets, where causally related events are ordered and concurrent events are unordered. Each causal poset describes a unique concurrent history, i.e., a set of executions, expressed as sequences or step sequences, that are consistent with it. Moreover, a poset captures all precedence-based invariant relationships between the events in the executions belonging to its concurrent history. However, concurrent histories in general may be too intricate to be described solely in terms of causal posets. In this paper, we introduce and investigate generalised mutex order structures which can capture the invariant causal relationships in *any* concurrent history consisting of step sequence executions. Each such structure comprises two relations, viz. interleaving/mutex and weak causality. As our main result we prove that each generalised mutex order structure is the intersection of the step sequence executions which are consistent with it.

Keywords: concurrent history, causal poset, weak causal order, mutex relation, interleaving, step sequence, causality semantics.

1. Introduction

In order to design and validate complex concurrent systems, it is essential to understand the fundamental relationships between the events occurring in their executions. However, sequential descriptions (specifications or observations) of executions are not sufficient when it comes to providing faithful information about causality and independence between events. To address this drawback, one may resort to using partially ordered sets of events to provide explicit representation of causality in the executions of a concurrent system. The order in which independent events are specified or observed may be accidental and those descriptions which only differ in the order of occurrences of independent events may be regarded as belonging to the same (*concurrent*) *history*, underpinned by a causal poset [1, 17, 20, 21, 26]. Note that

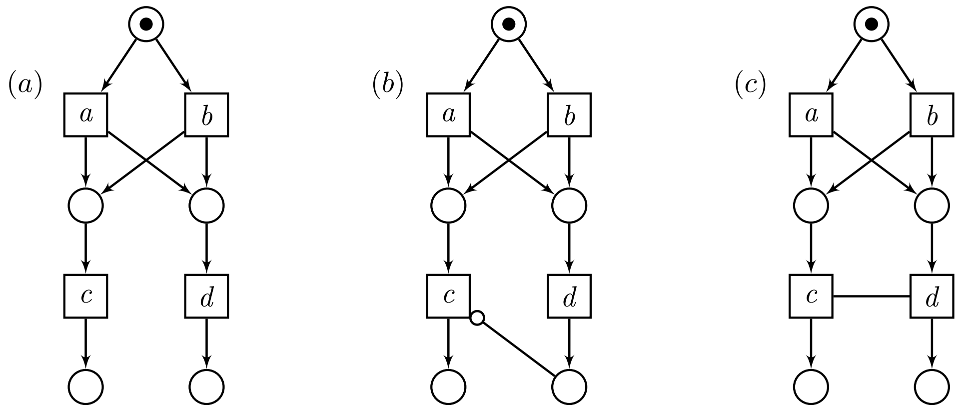

Figure 1. A safe Petri net (a), extended with an inhibitor arc implying that when c is executed the output place of d must be empty (b), and extended with a mutex arc implying that c and d cannot be executed simultaneously (c).

here executions are described as step sequences, i.e., sequences of finite sets (steps) of simultaneously executed events.

In general, concurrent behaviours can be investigated at the level of individual executions as well as at the level of order structures, like causal posets, capturing the essential invariant dependencies between events. The key link between these two levels is the notion of a *concurrent history* [9], an *invariant closed* set Δ (of descriptions) of executions. The latter means that Δ is fully determined by invariant relationships over X, its set of events: causality ($e \prec_\Delta f$ if, in all executions of Δ, e precedes f); weak causality ($e \sqsubset_\Delta f$ if, in all executions of Δ, e either precedes or is simultaneous with f); and interleaving/mutex ($e \rightleftharpoons_\Delta f$ if, in all executions of Δ, e is not simultaneous with f). In the case of safe Petri nets with sequential executions, $\prec_\Delta$ is the only invariant we need (as then, e.g., $\prec_\Delta = \sqsubset_\Delta$ and $\rightleftharpoons_\Delta = \prec_\Delta \cup \prec_\Delta^{-1}$). In particular, Δ is the set of all sequential executions corresponding to the linearisations of $\prec_\Delta$. The soundness of this approach is validated by Szpilrajn's Theorem [24] which states that each poset is equal to the intersection of its linearisations.

As an example, consider the safe Petri net depicted in Figure 1(a) which generates three step sequences involving a, c and d, viz. $\sigma = \{a\}\{c,d\}$, $\sigma' = \{a\}\{c\}\{d\}$ and $\sigma'' = \{a\}\{d\}\{c\}$. They can be seen as forming a single concurrent history $\Delta = \{\sigma, \sigma', \sigma''\}$ underpinned by a causal poset $\prec_\Delta$ satisfying $a \prec_\Delta c$ and $a \prec_\Delta d$. Moreover, this Δ adheres to the following *true concurrency paradigm*:

Given two events (c and d), they can be observed as simultaneous (in σ)

$\iff$

they can be observed in both orders (c before d in σ', and d before c in σ'').
(TRUECON)

Histories adhering to TRUECON are underpinned by *causal partial orders*, i.e., each such history comprises *all* step sequence executions consistent with a unique causal poset on events involved in the history.

In [9] fundamental concurrency paradigms are identified, including (TRUECON). Another paradigm, characterised by (TRUECON) with $\iff$ replaced by $\Longleftarrow$, has a natural interpretation provided by safe Petri nets with inhibitor arcs. Figure 1(b) depicts such a net generating two step sequences involving a, c

and d, viz. $\sigma = \{a\}\{c,d\}$ and $\sigma' = \{a\}\{c\}\{d\}$. They form a concurrent history $\Delta' = \{\sigma, \sigma'\}$ adhering to the paradigm that unorderedness implies simultaneity, but *not* to the true concurrency paradigm as Δ' has no step sequence in which d precedes c although c and d occur in a single step in σ.

As a result, histories adhering to this weaker paradigm are *not* underpinned by causal partial orders, but rather by causality structures $\langle X, \prec, \sqsubset \rangle$ introduced in [7] — called *stratified order structures* (SO-structures) — based on causality and an additional weak causality ('not later than') relation. A version of Szpilrajn's Theorem can be shown to hold also for SO-structures and the concurrent histories they generate. Stratified order structures were independently introduced in [3] (as 'prossets'). Their comprehensive theory was developed in e.g., [11, 12, 16, 19]. As shown in this paper, SO-structures are in a one-to-one correspondence with mutex order structures, or MO-structures, $\langle X, \rightleftharpoons, \sqsubset \rangle$ based on interleaving/mutex and weak causality. The first, symmetric, relation describes the events that only occur ordered (never simultaneously). Hence causality can be captured as a combination of mutex and weak causality.

This paper focuses on the least restrictive paradigm. i.e., there are no constraints imposed on concurrent histories. It admits all (invariant closed) concurrent histories comprising step sequence executions. As shown in [9], it is now sufficient to consider only two invariant relations, viz. mutex and weak causality. Figure 1(c) depicts a safe Petri net with mutex arcs (see [14]) generating two step sequences involving a, c and d, viz. $\sigma' = \{a\}\{c\}\{d\}$ and $\sigma'' = \{a\}\{d\}\{c\}$. We first observe that they form a concurrent history $\Delta'' = \{\sigma', \sigma''\}$ in which the executions of c and d interleave, and are both preceded by a; in other words, $c \rightleftharpoons_{\Delta''} d$, $a \sqsubset_{\Delta''} c$, $a \sqsubset_{\Delta''} d$ and $c \rightleftharpoons_{\Delta''} a \rightleftharpoons_{\Delta''} d$. That Δ'' is a concurrent history (i.e., it is invariant closed) then follows from the observation that Δ'' contains *all* step sequences involving a, c and d which obey these invariant relationships. However, Δ'' does *not* conform to the two earlier considered paradigms as there is no step sequence in Δ'' in which c and d occur simultaneously. To summarise, a nonempty set Δ of step sequence executions over a common set of events X, is a concurrent history iff Δ consists of all step sequences σ over X such that for all $e, f \in X$: $e \rightleftharpoons_\Delta f$ implies that e and f are not simultaneous in σ, and $e \sqsubset_\Delta f$ implies that e precedes or is simultaneous with f in σ.

The aim of this paper is to provide a structural characterisation of general concurrent histories (consisting of step sequence executions). An early attempt to describe structures of this kind was made in [4]. The there proposed generalised stratified order structures (or GSO-structures) do however not always capture all implied invariant relationships involving the mutex relation. Here, we will show that *generalised mutex order structures* (or GMO-structures) describe exactly all general concurrent histories. Our main result is a version of Szpilrajn's Theorem for GMO-structures and concurrent histories. For this we develop a notion of GMO-closure which corresponds to the transitive closure of an acyclic relation.

First, we recall key notions and notations used throughout the paper. In Section 3, we introduce MO-structures and establish their relationship with stratified order structures. Then, Section 4 introduces GMO-structures and proves their main properties, including GMO-closure and the GMO-structure version of Szpilrajn's Theorem. Section 5 presents concluding remarks.

A preliminary version of this paper without proofs appeared in [6].

2. Basic definitions

We use the standard notions of set theory and formal language theory. The identity relation on a set X is defined as $id_X = \{\langle a, a \rangle \mid a \in X\}$, the index X may be omitted if it is clear from the context.

2.1. Composing relations

The composition of two binary relations, R and Q, over X is given by

$$R \circ Q = \{\langle a,b\rangle \mid \exists c \in X :\ aRc \wedge cQb\}\ .$$

Moreover, if $P \subseteq X \times X$, then we define

$$R \circ_P Q = \{\langle a,b\rangle \mid \exists \langle x,y\rangle \in P :\ aRxQb\ \wedge\ aRyQb\}\ .$$

Note that $\circ = \circ_{id}$. The diagram below illustrates the derivation of $\langle a,b\rangle \in R \circ_P Q$:

Let $R \subseteq X \times X$, $R^0 = id$ and $R^n = R^{n-1} \circ R$, for all $n \geq 1$. Then: Then: (i) the reflexive closure of R is defined by $R \cup id$; (ii) the transitive closure by $R^+ = \bigcup_{i \geq 1} R^i$; (iii) the reflexive transitive closure by $R^* = id \cup R^+$; and (iv) the irreflexive transitive closure by $R^{\curlywedge} = R^+ \setminus id = R^* \setminus id$. Moreover, the inverse of R is given by $R^{-1} = \{\langle a,b\rangle \mid \langle b,a\rangle \in R\}$, and the symmetric closure by $R^{sym} = R \cup R^{-1}$.

We will denote $a_1 \ldots a_k R b_1 \ldots b_m$ whenever $a_i R b_j$, for all i, j. For example, $aRbcQd$ means that $aRbQd$ and $aRcQd$.

2.2. Order relations

A relation $R \subseteq X \times X$ is: (i) symmetric if $R = R^{-1}$; (ii) asymmetric if $R \cap R^{-1} = \varnothing$; (iii) reflexive if $id \subseteq R$; (iv) irreflexive if $id \cap R = \varnothing$; (v) transitive if $R \circ R \subseteq R$; and (vi) total if $R \cup R^{-1} = X \times X$.

A relation $R \subseteq X \times X$ is: (i) an *equivalence relation* if it is symmetric, transitive and reflexive; (ii) a *pre-order* if it is irreflexive and $R \cup id$ is transitive; (iii) a *partial order* if it is an asymmetric pre-order; and (iv) a *total order* if it is a partial order and $R \cup id$ is total; (v) a *stratified order* if it is a partial order such that $X \times X \setminus R^{sym}$ is an equivalence relation.

An irreflexive R induces a least pre-order containing R defined by $R^{\curlywedge}$. Following Schröder [23], $R^{\circledast} = R^* \cap (R^*)^{-1} = (R^{\curlywedge} \cap (R^{\curlywedge})^{-1}) \uplus id$ is the largest equivalence relation contained in R^*.

For a stratified order R we define two relations, $\sqsubset_R$ and $\rightleftharpoons_R$, such that, for all $a, b \in X$: $a \sqsubset_R b$ iff $a \neq b \wedge \neg(bRa)$, and $a \rightleftharpoons_R b$ iff $a \neq b \wedge \neg(a \sqsubset_R^{\circledast} b)$ (or iff $aRb \vee bRa$). If R represents a stratified order execution, aRb means 'a *occurred earlier than* b', $a \sqsubset_R b$ means 'a *occurred not later than* b', $a \rightleftharpoons_R b$ means 'a *did not occur simultaneously with* b', and $a \sqsubset_R^{\circledast} b$ means 'a *occurred simultaneously with* b'.

Directly from definitions we obtain a number of useful properties.

- Let R be a binary relation over X. Then, for all $a, b \in X$, we have:

$$(R \cup \langle a,b\rangle)^* = R^* \cup \{\langle c,d\rangle \mid cR^*a \wedge bR^*d\} \tag{1}$$

$$\neg(bR^*a) \implies (R \cup \langle a,b\rangle)^{\circledast} = R^{\circledast} \tag{2}$$

$$R^{\circledast} = (R^{\circledast})^{-1} \subseteq R^* \tag{3}$$

$$(R^{\curlywedge})^{\curlywedge} = R^{\curlywedge} \quad (R^{\curlywedge})^* = R^* \quad (R^*)^* = R^* \quad (R^{\curlywedge})^{\circledast} = R^{\circledast} \tag{4}$$

$$R^{\circledast} \circ R^{\circledast} = R^{\circledast} \quad R^* \circ R^{\circledast} = R^{\circledast} \circ R^* = R^* \circ R^* = R^* \tag{5}$$

- Let R be a stratified order over X. Then $\rightleftharpoons_R$ is irreflexive and symmetric, and $\sqsubset_R$ is a pre-order such that:

$$\sqsubset_R = \sqsubset_R^{+} \setminus id \; = \sqsubset_R^{\curlywedge} \quad \text{and} \quad \sqsubset_R^{\circledast} \setminus id \; = \sqsubset_R \cap \sqsubset_R^{-1} \; . \tag{6}$$

Moreover, for all distinct $a, b \in X$:

$$\begin{array}{lcl} \neg(a \rightleftharpoons_R b) & \Longleftrightarrow & \sqsubset_R b \wedge b \sqsubset_R a \\ \neg(a \sqsubset_R b) & \Longrightarrow & b \sqsubset_R a \end{array}$$

as well as

$$aRb \iff a \rightleftharpoons_R b \;\wedge\; a \sqsubset_R b \tag{7}$$

(thus '*a occurred earlier than b*' iff '*a and b were not simultaneous & a occurred not later than b*') and exactly one of the following holds:

$$\begin{array}{lcl} a \rightleftharpoons_R b \rightleftharpoons_R a \sqsubset_R b \not\sqsubset_R a & \text{or} & a \rightleftharpoons_R b \rightleftharpoons_R a \not\sqsubset_R b \sqsubset_R a \\ & \text{or} & a \not\rightleftharpoons_R b \not\rightleftharpoons_R a \sqsubset_R b \sqsubset_R a \; . \end{array} \tag{8}$$

2.3. Relational structures

A tuple $S = \langle X, R_1, R_2, \ldots, R_n \rangle$, where $n \geq 1$ and each $R_i \subseteq X \times X$ is a binary relation on X, is an (n-ary) *relational* structure. By the *domain* of a relational structure S we mean the set X. An *extension* of S is any relational structure $S' = \langle X, R'_1, R'_2, \ldots, R'_n \rangle$ satisfying $R_i \subseteq R'_i$, for every $1 \leq i \leq n$. We denote this by $S \subseteq S'$. The *intersection* of a nonempty family $\mathcal{R} = \{\langle X, R^i_1, \ldots, R^i_n \rangle \mid i \in I\}$ of relational structures with the same domain and arity is given by

$$\bigcap \mathcal{R} = \big\langle X, \bigcap_{i \in I} R^i_1, \ldots, \bigcap_{i \in I} R^i_n \big\rangle \; .$$

In what follows, we consider only relational structures that contain two relations and have finite domains.

A relational structure $S = \langle X, Q, R \rangle$ is *separable* if $Q \cap R^{\circledast} = \varnothing$, Q is symmetric and R is irreflexive. An intersection of separable relational structures with the same domain is also separable.

A relational structure $S = \langle X, Q, R \rangle$ is *saturated* in a family of relational structures $\mathscr{X}$ if it belongs to $\mathscr{X}$ and for every extension $S' \in \mathscr{X}$ of S, we have $S = S'$.

A *stratified order structure* (SO-structures) as defined and analysed in [8, 11] is a relational structure $sos = \langle X, \prec, \sqsubset \rangle$, where $\prec$ and $\sqsubset$ are binary relations over X such that, for all $a, b, c \in X$:

$$\begin{array}{ll} S1: & a \not\sqsubset a \\ S2: & a \prec b \implies a \sqsubset b \\ S3: & a \sqsubset b \sqsubset c \;\wedge\; a \neq c \implies a \sqsubset c \\ S4: & a \sqsubset b \prec c \;\vee\; a \prec b \sqsubset c \implies a \prec c \; . \end{array}$$

Intuitively, $a \prec b$ means '*a occurred earlier than b*', $a \sqsubset b$ means '*a occurred not later than b*', and $a \sqsubset b \sqsubset a$ means '*a occurred simultaneously with b*'. Note that an order relation R is stratified iff $\langle X, R, R \cup ((X \times X) \setminus (R \cup id)) \rangle$ is an SO-structure [8, 11]. Intuitively, stratified orders 'are' step sequences whereas stratified order structures can be saturated (extended) to define step sequences, similarly to extending partial orders to linear (total) orders.

A *generalized stratified order structure* (GSO-structure) is defined in [4] as a relational structure $gsos = \langle X, \rightleftharpoons, \sqsubset \rangle$ such that $\rightleftharpoons$ is irreflexive and symmetric, and $\langle X, \rightleftharpoons \cap \sqsubset, \sqsubset \rangle$ is an SO-structure. A comprehensive treatment of GSO-structures can be found in [5].

3. Order structures

In the rest of this paper, we will be concerned with relational structures of the form $S = \langle X, \rightleftharpoons, \sqsubset \rangle$. Intuitively, X is a set of events involved in some history of a concurrent system, $\rightleftharpoons$ is a 'mutex' (or 'interleaving') relation which relates pairs of events which cannot occur simultaneously, and $\sqsubset$ is a 'weak precedence' relation between events. The latter means, in particular, that if $a \sqsubset b \sqsubset a$ then a and b must occur simultaneously in any execution belonging to the history represented by S; i.e., S must be separable ($\rightleftharpoons \cap \sqsubset^{\circledast} = \varnothing$) as separability means that simultaneous events cannot be in the mutex relation.

Definition 3.1. (order structure)
An *order structure* is any separable relational structure $\langle X, \rightleftharpoons, \sqsubset \rangle$.

3.1. Mutex order structures

We now take a look at stratified order structures, in a way different from that of, e.g., [11, 16, 19]. More precisely, we will provide a new representation with causal order being replaced by a mutex relation. While SO-structures allow for a more compact representation (strict precedence involves fewer pairs of events than mutex), the new order structures are easier to generalise to cater for general interleaving/mutex requirements and their properties.

Definition 3.2. (mutex order structure)
A *mutex order structure* (MO-structure) is a relational structure $mos = \langle X, \rightleftharpoons, \sqsubset \rangle$, where $\rightleftharpoons$ and $\sqsubset$ are binary relations on X such that, for all $a, b, c \in X$:

$$\begin{array}{ll} \mathit{M1}: & a \rightleftharpoons b \implies b \rightleftharpoons a \\ \mathit{M2}: & a \not\sqsubset a \\ \mathit{M3}: & a \rightleftharpoons b \implies a \sqsubset b \vee b \sqsubset a \\ \mathit{M4}: & a \sqsubset b \sqsubset c \wedge a \neq c \implies a \sqsubset c \\ \mathit{M5}: & a \sqsubset b \sqsubset c \wedge (a \rightleftharpoons b \vee b \rightleftharpoons c) \implies a \rightleftharpoons c\,. \end{array}$$

Axioms $\mathit{M3} - \mathit{M5}$ are illustrated in Figure 2. Some properties of MO-structures are given below.

Proposition 3.1. Every MO-structure $mos = \langle X, \rightleftharpoons, \sqsubset \rangle$ is separable and, for all $a, b, c, d \in X$, we have:

(i) $a \not\rightleftharpoons a$;

(ii) if $a \sqsubset b \sqsubset a \rightleftharpoons c$ then $b \rightleftharpoons c$; and

(iii) if $a \sqsubset cd \sqsubset b$ and $c \rightleftharpoons d$ then $a \rightleftharpoons b$.

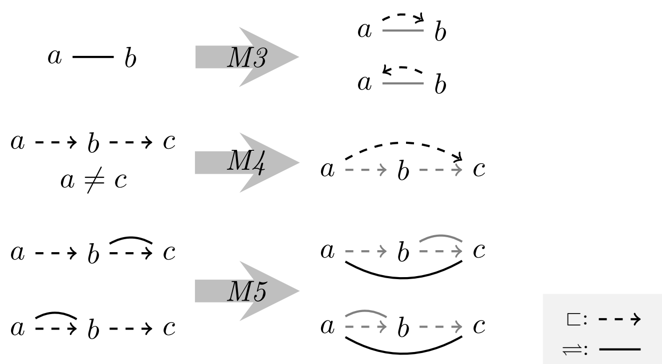

Figure 2. A visualisation of axioms $M3$, $M4$, and $M5$.

Proof:
Suppose that $a \sqsubset^* b \sqsubset^* a$ and $a \rightleftharpoons b$. Then $a \neq b$ and using $M4$ we may assume that $a \sqsubset b \sqsubset a$. By $M5$, we now have $a \rightleftharpoons a$, contradicting (i) proved below. Hence $a \sqsubset^* b \sqsubset^* a$ implies $a \not\rightleftharpoons b$. Moreover, $\rightleftharpoons$ is symmetric by $M1$, and $\sqsubset$ is irreflexive by $M2$.

To show (i), suppose that $a \rightleftharpoons a$. Then, by $M3$, we have $a \sqsubset a$, contradicting $M2$. Hence $a \not\rightleftharpoons a$.

To show (ii), suppose that $a \sqsubset b \sqsubset a$ and $a \rightleftharpoons c$. From $a \rightleftharpoons c$ and $M3$ it follows that $a \sqsubset c$ or $c \sqsubset a$. If $a \sqsubset c$ then $b \sqsubset a \sqsubset c$ and $a \rightleftharpoons c$, and so, by $M5$, we obtain $b \rightleftharpoons c$. Moreover, if $c \sqsubset a$ then $c \sqsubset a \sqsubset b$ and $a \rightleftharpoons c$, and so, by $M5$, we obtain $c \rightleftharpoons b$. Together with $M1$ this yields $b \rightleftharpoons c$.

To show (iii), suppose that $a \sqsubset cd \sqsubset b$ and $c \rightleftharpoons d$. From $c \rightleftharpoons d$ and $M3$ it follows that $c \sqsubset d$ or $d \sqsubset c$. Without loss of generality, we can assume that $c \sqsubset d$. Then $a \sqsubset c \sqsubset d$ and $c \rightleftharpoons d$, and so, by $M5$, we obtain $a \rightleftharpoons d$. Moreover, we have $a \sqsubset d \sqsubset b$, and so, also by $M5$, we obtain $a \rightleftharpoons b$. □

The next results demonstrate that MO-structures are in a one-to-one relationship with SO-structures. Below, we use two mappings: $\mathsf{so2mo}(sos) = \langle X, \prec^{sym}, \sqsubset\rangle$, for an SO-structure $sos = \langle X, \prec, \sqsubset\rangle$; and $\mathsf{mo2so}(mos) = \langle X, \rightleftharpoons \cap \sqsubset, \sqsubset\rangle$, for an MO-structure $mos = \langle X, \rightleftharpoons, \sqsubset\rangle$. Note that $\mathsf{so2mo}(sos)$ is obtained from sos by symmetrically closing the 'occurs earlier than' relation $\prec$ to reflect the fact that mutex is symmetric. This does not lead to a loss of any precedence relationships as $\prec$ is included in $\sqsubset$, by $S2$.

Theorem 3.1. The mappings mo2so and so2mo are inverse bijections.

Proof:
Suppose that $mos = \langle X, \rightleftharpoons, \sqsubset\rangle$ is an MO-structure. We start from showing that $\langle X, \rightleftharpoons \cap \sqsubset, \sqsubset\rangle$ is an SO-structure. Note that $S1 = M2$ and $S3 = M4$ because we use the same relation $\sqsubset$ in both cases. Hence it remains to be shown that $S2$ and $S4$ hold. Let $\prec$ denote $\rightleftharpoons \cap \sqsubset$.

$S2$ holds, since by $\prec = \rightleftharpoons \cap \sqsubset$, we have that $a \prec b$ implies $a \sqsubset b$.

To show $S4$, assume that $a \sqsubset b \prec c$ or $a \prec b \sqsubset c$. Suppose that $a = c$. Then, by Proposition 3.1(i), we have $a \not\rightleftharpoons c$. Also, we have $a \sqsubset b \sqsubset a$ and $a \rightleftharpoons b \vee b \rightleftharpoons a$. Hence, by $M5$, we obtain $a \rightleftharpoons c$, a contradiction. Hence $a \neq c$.

Let $a \prec b \sqsubset c$. Then $a \sqsubset b \sqsubset c$ and $a \rightleftharpoons b$. By $M5$, we obtain $a \rightleftharpoons c$. Since, by $M4$, we also have $a \sqsubset c$, it follows that $a \prec c$.
Let $a \sqsubset b \prec c$. Then $a \sqsubset b \sqsubset c$ and $b \rightleftharpoons c$. By $M5$, we obtain $a \rightleftharpoons c$. Since, by $M4$, we also have $a \sqsubset c$, it follows that $a \prec c$. Hence $\mathsf{mo2so}(mos)$ is an SO-structure.

Assume that $mos' = \langle X', \rightleftharpoons', \sqsubset' \rangle$ is an MO-structure such that $\mathsf{mo2so}(mos) = \mathsf{mo2so}(mos')$. Then, clearly, $X = X'$ and $\sqsubset \,=\, \sqsubset'$. Let $a \rightleftharpoons b$. Then, by $M3$ and without loss of generality, $a \sqsubset b$. Hence $\langle a, b \rangle \in \rightleftharpoons \cap \sqsubset$, and so $\langle a, b \rangle \in \rightleftharpoons' \cap \sqsubset'$. Thus $a \rightleftharpoons' b$. Conversely, we may show that $a \rightleftharpoons' b$ implies $a \rightleftharpoons b$. Hence $mos = mos'$, demonstrating that $\mathsf{mo2so}$ is injective.

Suppose now that $sos = \langle X, \prec, \sqsubset \rangle$ is an SO-structure. We will show that $\langle X, \prec^{sym}, \sqsubset \rangle$ is an MO-structure. Note that $M2 = S1$ and $M4 = S3$ because we use the same relation $\sqsubset$ in both cases. Hence it remains to be shown that $M1$, $M3$ and $M5$ hold. Let $\rightleftharpoons$ denote $\prec^{sym}$.

$M1$ clearly holds.

To show $M3$, we observe that $a \rightleftharpoons b$ implies $a \prec b \vee b \prec a$. Hence, by $S2$, we obtain $a \sqsubset b \vee b \sqsubset a$.

To show $M5$, suppose that $a \sqsubset b \sqsubset c$ and $a \rightleftharpoons b \vee b \rightleftharpoons c$. Then $a \sqsubset b \sqsubset c \wedge a \rightleftharpoons b$ or $a \sqsubset b \sqsubset c \wedge b \rightleftharpoons c$. Hence, by $\rightleftharpoons \,=\, \prec^{sym}$, we obtain $a \sqsubset b \sqsubset c$ and

$$a \prec b \vee b \prec a \vee b \prec c \vee c \prec b \;.$$

Since $a \sqsubset b$ implies $\neg(b \prec a)$ by $S4, S2, S1$, we can exclude $b \prec a$ and $c \prec b$, and so we have that $a \sqsubset b \prec c$ or $a \prec b \sqsubset c$. Hence, using $S4$, we get $a \prec c$. Once more using the definition of $\rightleftharpoons$, we obtain $a \rightleftharpoons c$. Hence $\mathsf{so2mo}(sos)$ is an MO-structure.

Assume that $sos' = \langle X', \prec', \sqsubset' \rangle$ is an SO-structure such that $\mathsf{so2mo}(sos) = \mathsf{so2mo}(sos')$. Then, clearly, $X = X'$ and $\sqsubset = \sqsubset'$. Let $a \prec b$. Then, by $S2$, $a \sqsubset b$ and so $a \sqsubset' b$. Moreover, $\langle a, b \rangle \in \prec^{sym} = (\prec')^{sym}$. If $\langle a, b \rangle \in (\prec')^{-1}$ then, by $S4$, $a \prec' a$. In this way, we obtained a contradiction with $S1$ and $S2$. Thus $a \prec' b$. Conversely, we may show that $a \prec' b$ implies $a \prec b$. Hence $mos = mos'$, and so $\mathsf{so2mo}$ is injective.

To show that $\mathsf{so2mo}$ and $\mathsf{mo2so}$ are inverse mappings, we observe, for all $a \neq b \in X$:

$$\langle a, b \rangle \in (\rightleftharpoons \cap \sqsubset)^{sym} \quad \text{iff} \quad a \rightleftharpoons b \wedge (a \sqsubset b \vee b \sqsubset a) \quad \text{iff (by } M3) \quad a \rightleftharpoons b \;.$$

Thus

$$\mathsf{so2mo} \circ \mathsf{mo2so}(X, \rightleftharpoons, \sqsubset) = \mathsf{so2mo}(X, \rightleftharpoons \cap \sqsubset, \sqsubset) = \langle X, (\rightleftharpoons \cap \sqsubset)^{sym}, \sqsubset \rangle = \langle X, \rightleftharpoons, \sqsubset \rangle \;.$$

Moreover,

$$\langle a, b \rangle \in \prec^{sym} \cap \sqsubset \quad \text{iff} \quad a \prec b \vee b \prec a \sqsubset b \quad \text{iff} \quad a \prec b$$

since, by $S4$, $b \prec a \sqsubset b$ implies $b \prec b$, contradicting $S1$ and $S2$. Hence

$$\mathsf{mo2so} \circ \mathsf{so2mo}(X, \prec, \sqsubset) = \mathsf{mo2so}(X, \prec^{sym}, \sqsubset) = \langle X, \prec^{sym} \cap \sqsubset, \sqsubset \rangle = \langle X, \prec, \sqsubset \rangle \;,$$

which completes the proof. □

3.2. Layered order structures

In general, order structures are not saturated and thus represent several executions just like a single partial order may have several total order linearisations. We will now define order structures that are in a one-to-one relationship with step sequences.

Definition 3.3. A relational structure $los = \langle X, \rightleftharpoons, \sqsubset \rangle$ is a *layered order structure* (LO-structure) if there exists a stratified order R over X such that $\rightleftharpoons = \rightleftharpoons_R$ and $\sqsubset = \sqsubset_R$.

First we show that LO-structures are MO-structures, and hence correspond to SO-structures.

Proposition 3.2. Every LO-structure $los = \langle X, \rightleftharpoons, \sqsubset \rangle$ is an MO-structure.

Proof:
To show $M1$, we observe that since $\rightleftharpoons$ is symmetric, we have $a \rightleftharpoons b \Longrightarrow b \rightleftharpoons a$.

To show $M2$, we observe that by Definition 3.3, $\sqsubset$ is irreflexive. Hence $a \not\sqsubset a$.

To show $M3$, suppose that $a \rightleftharpoons b$. Then, by (8), we get that $a \sqsubset b \wedge b \not\sqsubset a$ or $a \not\sqsubset b \wedge b \sqsubset a$. Hence $a \sqsubset b$ or $b \sqsubset a$.

To show $M4$, suppose that $a \sqsubset b \sqsubset c$ and $a \neq c$. Then $a \sqsubset^{\curlywedge} c$. Hence, by transitivity of $\sqsubset$, we obtain $a \sqsubset c$.

To show $M5$, let first $a \sqsubset b \sqsubset c$ and $a \rightleftharpoons b$. Suppose that $a = c$. Then $a \sqsubset b \sqsubset a$ and $a \rightleftharpoons b$, contradicting (8). Hence $a \neq c$. Thus, by $M4$, which we already proved, it follows from $a \sqsubset b \sqsubset c$ that $a \sqsubset c$. Moreover, $G2$ implies that $a \neq b$. In case $c \sqsubset a$, we have $a \sqsubset b \sqsubset c \sqsubset a$ and thus again by $M4$ $a \sqsubset b \sqsubset a$ which in combination with $a \rightleftharpoons b$ leads to a contradiction with (8). Hence $c \not\sqsubset a$ and thus, by (8), $a \rightleftharpoons c$. Now assume $a \sqsubset b \sqsubset c$ and $b \rightleftharpoons c$. As before, we must have $a \neq c$, $a \sqsubset b \sqsubset c$, and $b \neq c$. If $c \sqsubset a$, then $c \sqsubset b \sqsubset c$ holds which in combination with $b \rightleftharpoons c$ contradicts (8). Hence $c \not\sqsubset a$ and thus, by (8), also in this case $a \rightleftharpoons c$ which completes the proof. □

We can further show that LO-structures are indeed saturated.

Proposition 3.3. Every LO-structure is an order structure, and saturated in the set of all order structures.

Proof:
The first part follows from Propositions 3.2 and 3.1.

Let $S = \langle X, Q, R \rangle$ be an order structure extending los. Suppose that aQb and $a \not\rightleftharpoons b$. Then, by (8), we get $a \sqsubset b$ and $b \sqsubset a$. Hence aRb and bRa, and so $aR^{\circledast}b$. As a result, we obtain $\langle a, b \rangle \in Q \cap R^{\circledast}$ which contradicts the separability of S. This means that Q is equal to $\rightleftharpoons$.
Suppose now that aRb and $a \not\sqsubset b$. Then, by (8), we get $b \rightleftharpoons a$ and $b \sqsubset a$. Hence bQa and bRa which, together with aRb, gives $aR^{\circledast}b$. As a result, we obtain $\langle a, b \rangle \in Q \cap R^{\circledast}$ which contradicts the separability of S. This means that R is equal to $\sqsubset$, completing the proof. □

For an order structure $os = \langle X, \rightleftharpoons, \sqsubset \rangle$, we will denote by $\mathsf{os2los}(os)$ the set of all LO-structures los extending os, i.e., $os \subseteq los$. With this notation, a nonempty set LOS of LO-structures with the same domain is a concurrent history if

$$LOS = \mathsf{os2los}\big(\bigcap LOS\big)\ .$$

An MO-structure is linked with LO-structures (step sequences) through the set $\mathsf{os2los}(mos)$ of all LO-structures extending mos. Similarly, we can define $\mathsf{so2los}(sos) = \mathsf{os2los}(\mathsf{so2mo}(sos))$, for every SO-structure sos. It can then be seen ([11]) that $\mathsf{so2los}(sos) \neq \varnothing$ and (in the notation used in this paper)

$$sos = \bigcap \mathsf{mo2so}(\mathsf{so2los}(sos))\ .$$

That result corresponds to Szpilrajn's Theorem that every partial order is the intersection of its linearisations (see [5, 11]). This result extends to MO-structures and we obtain

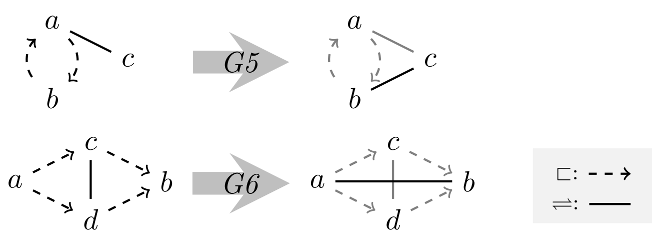

Figure 3. A visualisation of axioms $G5$ and $G6$.

Theorem 3.2. For every MO-structure mos,

$$\mathsf{os2los}(mos) \neq \varnothing \text{ and } mos = \bigcap \mathsf{os2los}(mos)\ .$$

We can therefore conclude that the saturated extensions of an MO-structure mos form a concurrent history represented by mos. It is then important to ask which concurrent histories can be derived in this way; in other words, which concurrent histories can be represented by MO-structures.

Consider now a nonempty set $LOS = \{\langle X, \rightleftharpoons_i, \sqsubset_i\rangle \mid i \in I\}$ of LO-structures forming a concurrent history, and their intersection $S = \bigcap LOS = \langle X, \rightleftharpoons, \sqsubset\rangle$. Since every LO-structure is also an MO-structure, we immediately obtain that S is an order structure satisfying axioms $M1$, $M2$, $M4$ and $M5$. However, $M3$ in general does not hold although it holds for histories in which the possibility of executing two events in either order implies also simultaneous execution, meaning that, for all distinct $a, b \in X$,

$$\Big(\exists i \in I : \langle a, b\rangle \in \rightleftharpoons_i \cap \sqsubset_i \quad \wedge \quad \exists j \in I : \langle b, a\rangle \in \rightleftharpoons_j \cap \sqsubset_j\Big) \implies \exists k \in I : \langle a, b\rangle \in \sqsubset_k^{sym}\ .$$

One might now wonder what happens if we do not assume any special properties of a concurrent history. As we will show in the rest of the paper, Proposition 3.1 in combination with the observation that it always holds for $S = \bigcap LOS$, yields a characterisation for the order structures underpinning general histories.

4. Generalised order structures

In this section, we provide a complete characterisation of general concurrent histories where executions are represented by layered order structures; i.e., a characterisation of concurrent histories comprising step sequence executions. We achieve this by retaining all those MO-structure axioms which hold in general, and then replacing the only dropped axiom $M3$ by Proposition 3.1.

Definition 4.1. (generalised mutex order structure)
A *generalised mutex order structure* (GMO-structure) is a relational structure $gmos = \langle X, \rightleftharpoons, \sqsubset\rangle$, where

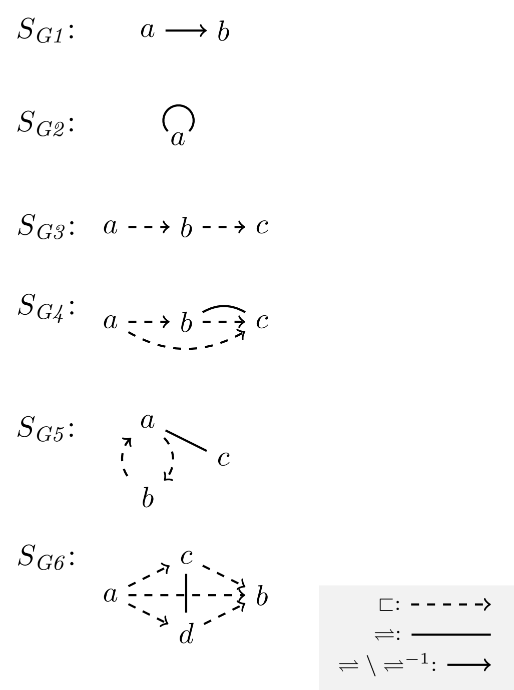

Figure 4. The set of axioms in Definition 4.1 is minimal: each S_{Gi} satisfies all the axioms except for Gi.

$\rightleftharpoons$ and $\sqsubset$ are binary relations on X such that, for all $a, b, c, d \in X$:

$G1$	:	$a \rightleftharpoons b \implies b \rightleftharpoons a$	$M1$
$G2$	:	$a \not\sqsubset a \;\wedge\; a \not\rightleftharpoons a$	$M2$ & $Prop.3.1(i)$
$G3$	:	$a \sqsubset b \sqsubset c \wedge a \neq c \implies a \sqsubset c$	$M4$
$G4$	:	$a \sqsubset b \sqsubset c \wedge (a \rightleftharpoons b \vee b \rightleftharpoons c) \implies a \rightleftharpoons c$	$M5$
$G5$	:	$a \sqsubset b \sqsubset a \wedge a \rightleftharpoons c \implies b \rightleftharpoons c$	$Prop.3.1(ii)$
$G6$	:	$a \sqsubset c \sqsubset b \wedge a \sqsubset d \sqsubset b \wedge c \rightleftharpoons d \implies a \rightleftharpoons b$	$Prop.3.1(iii)$

Axioms $G5$ and $G6$ are illustrated in Figure 3. We also note that the set of axioms in Definition 4.1 is minimal (see Figure 4). In fact, it remains minimal even if one splits $G2$ into two axioms (a suitable example here would be a structure with a single element a satisfying only $a \sqsubset a$). Moreover, we have the following.

Proposition 4.1. Let $gmos = \langle X, \rightleftharpoons, \sqsubset \rangle$ be a GMO-structure. Then $gmos$ is separable and, for all $a, b \in X$:

(i) $a \sqsubset^{\curlywedge} b$ implies $a \sqsubset b$; and

(ii) $a \sqsubset b \sqsubset a$ implies $a \not\rightleftharpoons b$.

Proof:
By (i, ii) proved below and axiom $G2$, we get $\rightleftharpoons \cap \sqsubset^{\circledast} = \varnothing$. The irreflexivity of $\sqsubset$ follows from axiom $G2$, while symmetry of $\rightleftharpoons$ from axiom $G1$.

If $a \sqsubset^{\curlywedge} b$ then there exists a sequence $c_1, \ldots, c_n$ such that $a = c_1$ and $b = c_n$ and, for $i = 1, \ldots, n-1$, we have $c_i \sqsubset c_{i+1}$. Let $c_1, \ldots, c_n$ be the shortest such sequence. Then $i \neq j$ implies $c_i \neq c_j$. Hence we can $n-2$ times use axiom $G3$ and obtain $a \sqsubset b$. Moreover, each GMO-structure satisfies axioms $G4$ and $G2$. Hence, as in the proof of Proposition 3.1, we obtain $a \sqsubset b \sqsubset a \implies a \neq b$. □

Proposition 4.2. Every MO-structure is a GMO-structure.

Proof:
Note that axioms $M1$, $M4$ and $M5$ are equivalent to axioms $G1$, $G3$ and $G4$, respectively. Moreover, by Proposition 3.1 and axiom $M2$, every MO-structure satisfies also axioms $G2$, $G5$ and $G6$. □

The converse of Proposition 4.2 does not hold; e.g., as $M3$ does not hold, $\langle\{a, b\}, \{\langle a, b\rangle, \langle b, a\rangle\}, \varnothing\rangle$ is a GMO-structure but *not* an MO-structure.

Proposition 4.3. If $gmos = \langle X, \rightleftharpoons, \sqsubset\rangle$ is a GMO-structure, then $\langle X, \rightleftharpoons \cap \sqsubset, \sqsubset\rangle$ is an SO-structure.

Proof:
Let $\prec$ denote $\rightleftharpoons \cap \sqsubset$. Only $S4$ is not obvious. Assume $a \sqsubset b \prec c$. From $G4$ we have $a \rightleftharpoons c$. Thus $a \neq c$, and so from $G3$ we have $a \sqsubset c$. Hence $a \prec c$. Now assume $a \prec b \sqsubset c$. If $a = c$, then $a \sqsubset b \sqsubset a$ and $a \rightleftharpoons b$. So $a \rightleftharpoons a$ by $G4$, a contradiction with $G2$. Hence $a \neq c$. From $a \sqsubset b \sqsubset c$ and $a \rightleftharpoons b$ it now follows that $a \rightleftharpoons c$ which together with $a \sqsubset c$ implies that $a \prec c$. □

Proposition 4.3 states that every GMO-structure is a GSO-structure. We observe that the converse is not true, with suitable counterexamples provided by the GSO-structures S_{G5} and S_{G6} in Figure 4.

4.1. Closure operator for generalised mutex order structures

We will now provide a method for deriving valid GMO-structures from other relational structures.

Definition 4.2. (GMO-closure)
Let $S = \langle X, Q, R\rangle$ be a relational structure and $Q^{[R]} = R^{\circledast} \circ \left\langle Q \cup (R^* \circ_Q R^*)^{sym}\right\rangle \circ R^{\circledast}$. Then the GMO-closure of S is given by

$$S^{\blacklozenge} = \left\langle X \;,\; Q^{[R]} \;,\; R^{\curlywedge}\right\rangle .$$

The GMO-closure operator can be seen as related to the transitive closure operator of acyclic reflexive binary relations, as well as to the closure of acyclic relational structures investigated in [10] in order to obtain SO-structures, and also to the closure operator introduced in [13] in order to obtain GSO-structures.

The property we need is that whenever $S = \langle X, Q, R\rangle$ is separable, $S^{\blacklozenge}$ is a GMO-structure. Furthermore, if S is already a GMO-structure, then we want $S^{\blacklozenge} = S$. The form of $Q^{[R]}$ follows from the requirement that $S^{\blacklozenge}$ should be a GMO-structure and the axioms for GMO-structures (see also Figure 5). In particular the factor $(R^* \circ_Q R^*)^{sym}$ follows from axioms $G4$ and $G6$, while the factor $R^{\circledast} \circ_Q R^{\circledast}$ corresponds to $G5$.

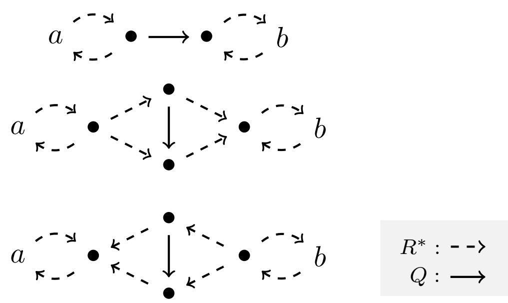

Figure 5. A visualisation of the three cases of $\langle a, b\rangle \in Q^{[R]}$.

The next set of results correspond to saying that: the transitive closure of an acyclic relation is also acyclic (Proposition 4.4(i)); GMO-closure is a closure operation in the usual sense (Proposition 4.4(ii)); the transitive closure of an acyclic relation yields a poset (Proposition 4.4(iii)); and posets are transitively closed (Proposition 4.4(ii)). First, we give a technical lemma.

Lemma 4.1. If $S = \langle X, Q, R\rangle$ is a relational structure, then

$$R^* \circ_{Q^{[R]}} R^* \subseteq R^* \circ_Q R^* \ .$$

Proof:
Suppose $\langle a, b\rangle \in R^* \circ_{Q^{[R]}} R^*$. Then there is $\langle c, d\rangle \in Q^{[R]}$ such that aR^*cdR^*b. This is equivalent to saying that there are $c, d, e, f \in X$ such that aR^*cdR^*b and $cR^{\circledast}eZfR^{\circledast}d$, where

$$Z = Q \cup (R^* \circ_Q R^*) \cup (R^* \circ_Q R^*)^{-1} \ .$$

Thus, by (3) and (5), aR^*cdefR^*b (†). We then consider three cases corresponding to three parts of Z from which the relationship between e and f has been derived (see Figures 5 and 6).

Case 1: $\langle e, f\rangle \in Q$. Then, by (†), aR^*efR^*b. Hence $\langle a, b\rangle \in R^* \circ_Q R^*$.

Case 2: $\langle e, f\rangle \in R^* \circ_Q R^*$. Then there is $\langle g, h\rangle \in Q$ such that eR^*ghR^*f. Thus, by (†) and (5), we have aR^*ghR^*b. Hence $\langle a, b\rangle \in R^* \circ_Q R^*$.

Case 3: $\langle e, f\rangle \in (R^* \circ_Q R^*)^{-1}$. Then there is $\langle g, h\rangle \in Q$ such that fR^*ghR^*e. Thus, by (†) and (5), we have aR^*ghR^*b. Hence $\langle a, b\rangle \in R^* \circ_Q R^*$. □

Proposition 4.4. Let $S = \langle X, Q, R\rangle$ be an order structure. then:

(i) $S^{\blacklozenge}$ is an order structure.

(ii) $S \subseteq S^{\blacklozenge}$ and $(S^{\blacklozenge})^{\blacklozenge} = S^{\blacklozenge}$.

(iii) $S^{\blacklozenge}$ is a GMO-structure.

Proof:
To show (i) we prove separability. We first note that $R^{\circledast}$ is symmetric. Since a composition of symmetric relations is symmetric, we have that $Q^{[R]}$ is symmetric. Moreover, $R^{\curlywedge}$ is irreflexive by definition.

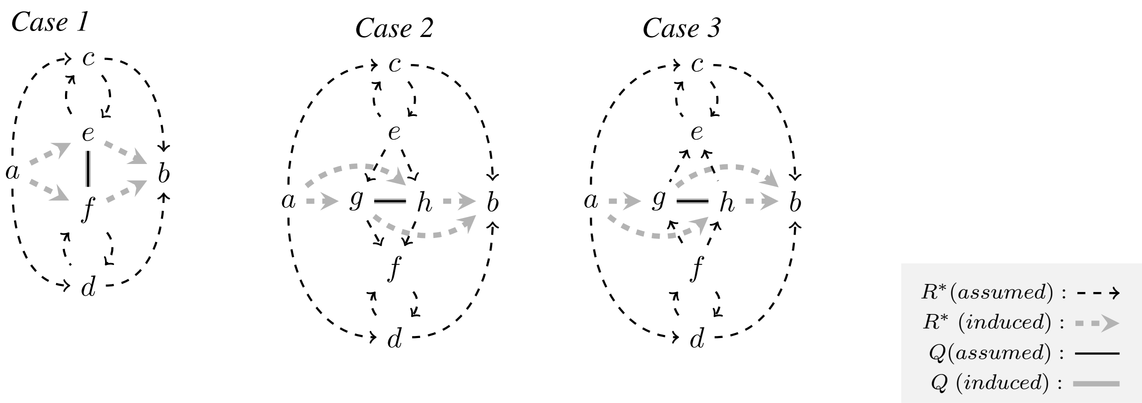

Figure 6. A visualisation of the proof of Lemma 4.1.

To prove that $Q^{[R]} \cap (R^{\curlywedge})^{\circledast} = \varnothing$, by $(R^{\curlywedge})^{\circledast} = R^{\circledast}$, it suffices to show that $Q^{[R]} \cap R^{\circledast} = \varnothing$. Suppose that $\langle a, b\rangle \in Q^{[R]} \cap R^{\circledast}$. By $\langle a, b\rangle \in Q^{[R]}$, there are $c, d \in X$ such that $aR^{\circledast}cZdR^{\circledast}b$, where $Z = Q \cup (R^* \circ_Q R^*) \cup (R^* \circ_Q R^*)^{-1}$. We consider three cases (see Figure 7).

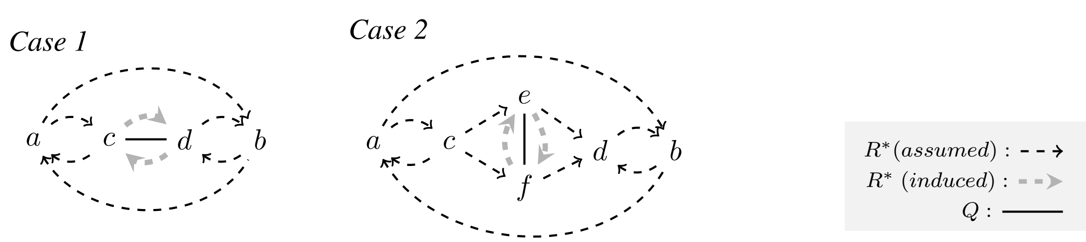

Figure 7. A visualisation of the proof of Proposition 4.4(i).

Case 1: $\langle c, d\rangle \in Q$. Then $cR^{\circledast}aR^{\circledast}bR^{\circledast}d$, so by (5), $\langle c, d\rangle \in R^{\circledast}$, contradicting the separability of S.

Case 2: $\langle c, d\rangle \in R^* \circ_Q R^*$. Then there is $\langle e, f\rangle \in Q$ such that cR^*efR^*d. Hence $\langle e, f\rangle \in R^{\circledast}$, by

$$eR^*dR^{\circledast}bR^{\circledast}aR^{\circledast}cR^*fR^*dR^{\circledast}bR^{\circledast}aR^{\circledast}cR^*e\ .$$

This, however, contradicts the separability of S.

Case 3: $\langle c, d\rangle \in (R^* \circ_Q R^*)^{-1}$. Similar to Case 2.

To show (ii), we observe that the first part follows from the reflexivity of $R^{\circledast}$ and the irreflexivity of R. From (4) it follows that to prove the second part, it suffices to establish that $(Q^{[R]})^{[R^{\curlywedge}]} = Q^{[R]}$. Note that $Q^{[R]} \subseteq (Q^{[R]})^{[R^{\curlywedge}]}$ by $S \subseteq S^{\blacklozenge}$ and by (ii) both just proved. To prove the converse inclusion we proceed as follows.

We have $(Q^{[R]})^{[R^{\curlywedge}]} = R^{\circledast} \circ \big(Q^{[R]} \cup (R^* \circ_{Q^{[R]}} R^*)^{sym}\big) \circ R^{\circledast}$, by definition.

Then, from the definition of $Q^{[R]}$ and (5) we obtain:

$$\begin{aligned}
&R^{\circledast} \circ \big(Q^{[R]} \cup (R^{*} \circ_{Q^{[R]}} R^{*})^{sym}\big) \circ R^{\circledast} \\
&= R^{\circledast} \circ \big((R^{\circledast} \circ (Q \cup (R^{*} \circ_{Q} R^{*})^{sym}) \circ R^{\circledast}) \cup (R^{*} \circ_{Q^{[R]}} R^{*})^{sym}\big) \circ R^{\circledast} \\
&= R^{\circledast} \circ R^{\circledast} \circ Q \circ R^{\circledast} \circ R^{\circledast} \;\cup\; R^{\circledast} \circ R^{\circledast} \circ (R^{*} \circ_{Q} R^{*})^{sym} \circ R^{\circledast} \circ R^{\circledast} \\
&\qquad \cup\; R^{\circledast} \circ (R^{*} \circ_{Q^{[R]}} R^{*})^{sym} \circ R^{\circledast} \\
&= R^{\circledast} \circ Q \circ R^{\circledast} \;\cup\; R^{\circledast} \circ (R^{*} \circ_{Q} R^{*})^{sym} \circ R^{\circledast} \;\cup\; R^{\circledast} \circ (R^{*} \circ_{Q^{[R]}} R^{*})^{sym} \circ R^{\circledast} \;.
\end{aligned}$$

Next, by Lemma 4.1 and again the definition of $Q^{[R]}$ and (5), we conclude

$$\begin{aligned}
&R^{\circledast} \circ Q \circ R^{\circledast} \;\cup\; R^{\circledast} \circ (R^{*} \circ_{Q} R^{*})^{sym} \circ R^{\circledast} \;\cup\; R^{\circledast} \circ (R^{*} \circ_{Q^{[R]}} R^{*})^{sym} \circ R^{\circledast} \\
&\subseteq R^{\circledast} \circ Q \circ R^{\circledast} \;\cup\; R^{\circledast} \circ (R^{*} \circ_{Q} R^{*})^{sym} \circ R^{\circledast} \;\cup R^{\circledast} \circ (R^{*} \circ_{Q} R^{*})^{sym} \circ R^{\circledast} \\
&= R^{\circledast} \circ Q \circ R^{\circledast} \;\cup\; R^{\circledast} \circ (R^{*} \circ_{Q} R^{*})^{sym} \circ R^{\circledast} \\
&= Q^{[R]} \;.
\end{aligned}$$

Hence $(Q^{[R]})^{[R^{\curlywedge}]} \subseteq Q^{[R]}$, and so $(Q^{[R]})^{[R^{\curlywedge}]} = Q^{[R]}$.

To show (iii), by (i) and (ii), it suffices to show that $S^{\blacklozenge}$ satisfies all the axioms in Definition 4.1 in the case that $S^{\blacklozenge} = S$, i.e., $Q = Q^{[R]}$ and $R = R^{\curlywedge}$.
To show $G1$, we observe that, by (i), Q is symmetric.
To show $G2$, we first observe that, by definition, $\langle a, a\rangle \notin R^{\curlywedge} = R$. Suppose that $\langle a, a\rangle \in Q$. Then, since $\langle a, a\rangle \in R^{\circledast}$, we have $\langle a, a\rangle \in Q^{[R]} \cap (R^{\curlywedge})^{\circledast}$, contradicting (i).
To show $G3$, suppose that $aRbRc$ and $a \neq c$. Then $\langle a, c\rangle \in R^{\curlywedge} = R$.
To show $G4$, suppose that $aRbRc$ and aQb. If $a = c$ then, by the separability of S, $\langle a, b\rangle \notin Q$, a contradiction. Hence, by $G3$ (already shown), we have aRc. Thus $aR^{*}abR^{*}c$, and so $\langle a, c\rangle \in R^{*} \circ_{Q} R^{*} \subseteq Q^{[R]} = Q$. If $aRbRc$ and bQc, we proceed similarly using $aR^{*}bcR^{*}c$.
To show $G5$, suppose that $aRbRa$ and aQc. Since $\langle c, c\rangle \in R^{\circledast}$, we obtain $\langle b, c\rangle \in R^{\circledast} \circ Q \circ R^{\circledast}$, and so $\langle b, c\rangle \in Q^{[R]} = Q$.
To show $G6$, suppose $aRcdRb$ and cQd. Then $\langle a, b\rangle \in R^{*} \circ_{Q} R^{*} \subseteq Q^{[R]} = Q$. □

Proposition 4.5. If $gmos = \langle X, \rightleftharpoons, \sqsubset\rangle$ is a GMO-structure, then $gmos^{\blacklozenge} = gmos$.

Proof:
By Proposition 4.1 we get that $gmos$ is separable. Moreover, $\sqsubset \subseteq \sqsubset^{\curlywedge}$ and $a \sqsubset^{\circledast} b$ iff $a \sqsubset b \sqsubset a \vee a = b$.

To show that $\rightleftharpoons$ is equal to $\rightleftharpoons^{[\sqsubset]}$, we first observe that, by Definition 4.2 and the reflexivity of $\sqsubset^{\circledast}$, we have that $\rightleftharpoons$ is contained in $\rightleftharpoons^{[\sqsubset]}$. To show that $\rightleftharpoons^{[\sqsubset]}$ is contained in $\rightleftharpoons$, suppose that $a \rightleftharpoons^{[\sqsubset]} b$, which means that there are $c, d \in X$ such that:

(i) $a \sqsubset c \sqsubset a \vee a = c$ and $b \sqsubset d \sqsubset b \vee b = d$;

and one of the following is satisfied:

(ii) $a \sqsubset^{\circledast} c \rightleftharpoons d \sqsubset^{\circledast} b$; or

(iii) $a \sqsubset^{\circledast} c(\sqsubset^{*} \circ_{\rightleftharpoons} \sqsubset^{*})d \sqsubset^{\circledast} b$; or

(iv) $a \sqsubset^{\circledast} c(\sqsubset^{*} \circ_{\rightleftharpoons} \sqsubset^{*})^{-1} d \sqsubset^{\circledast} b$.

If (ii) holds then, by (i) and $G5$, we get $a \rightleftharpoons d$. Hence, by $G1$, $d \rightleftharpoons a$. Therefore, by (i) and $G5$, $b \rightleftharpoons a$. Hence, by $G1$, we obtain $a \rightleftharpoons b$.

If (iii) holds, then there are $e \rightleftharpoons f$ such that $c \sqsubset^{*} ef \sqsubset^{*} d$. Hence $a \sqsubset^{*} ef \sqsubset^{*} b$. By Proposition 4.1, we need to consider sixteen different cases, as $x \sqsubset^{*} y$ is equivalent to $x \sqsubset y \ \vee \ x = y$. Most of them may be excluded, as the roles of e and f are symmetric and, by $G2$, we have $e \neq f$. Moreover, $a \neq b$ follows from the fact that $gmos^{\blacklozenge}$ is a GMO-structure by Proposition 4.4(iii) and $G2$. Hence, together with $e \rightleftharpoons f$, we get a contradiction with the separability of $gmos$. As a result we have to consider only four cases.

Case 1: $a = e$ and $b = f$. Then $a \rightleftharpoons b$.

Case 2: $a = e$ and $b \neq f$. Then $a \sqsubset f \sqsubset b$ and $a \rightleftharpoons f$. Hence, by $G4$, $a \rightleftharpoons b$.

Case 3: $a \neq e$ and $b = f$. Then $a \sqsubset e \sqsubset b$ and $e \rightleftharpoons b$. Hence, by $G4$, $a \rightleftharpoons b$.

Case 4: a, b, e and f are all distinct. Then $a \sqsubset ef \sqsubset b$. Hence, by $G6$, $a \rightleftharpoons b$.

Finally, if (iv) holds, then $\langle b, a\rangle \in \sqsubset^{\circledast} \circ (\sqsubset^{*} \circ_{\rightleftharpoons} \sqsubset^{*}) \circ \sqsubset^{\circledast}$, as $\sqsubset^{\circledast}$ is symmetric. Hence, by (iii), we get $b \rightleftharpoons a$. Thus, by $G1$, we obtain $a \rightleftharpoons b$. □

As layered order structures and mutex order structures are special cases of generalised mutex order structures, we obtain an immediate

Corollary 4.1. If los is an LO-structure and mos an MO-structure, then $los^{\blacklozenge} = los$ and $mos^{\blacklozenge} = mos$.

The following technical lemma describes a single stage of the saturation process for a GMO-structure leading to a set of LO-structures. In this process, we may add an arbitrary link between two elements that do not yet satisfy (8). We only need to remember that in the case of extending the relation Q, together with $\langle a, b\rangle$ we have to add $\langle b, a\rangle$. After this addition, we get an order structure that may be closed. As a result, we obtain one of the possible extensions of an initial $gmos$. The above process terminates after obtaining an LO-structure and it is central to the proof of the main Theorem 4.1.

In what follows, we denote $R_{xy} = R \cup \{\langle x, y\rangle\}$ and $Q_{xyx} = Q \cup \{\langle x, y\rangle, \langle y, x\rangle\}$.

Lemma 4.2. Let $gmos = \langle X, Q, R\rangle$ be a GMO-structure, $a, b \in X$ and $a \neq b$.

(i) If $\langle a, b\rangle \notin R$ and $\langle b, a\rangle \notin R$ then $\langle X, Q, R_{ab}\rangle^{\blacklozenge}$ is a GMO-structure.

(ii) If $\langle a, b\rangle \notin R$ and $\langle a, b\rangle \notin Q$ then $\langle X, Q, R_{ab}\rangle^{\blacklozenge}$ is a GMO-structure.

(iii) If $\langle a, b\rangle \notin R$ and $\langle a, b\rangle \notin Q$ then $\langle X, Q_{aba}, R\rangle^{\blacklozenge}$ is a GMO-structure.

Proof:
By Proposition 4.5, $gmos^{\blacklozenge} = gmos$, hence $R = R^{\curlywedge}$ and $Q = Q^{[R]}$. To obtain the thesis, it suffices to prove the separability of enriched structures:

(i') $\langle a, b\rangle \notin R \wedge \langle b, a\rangle \notin R$ implies $Q \cap R_{ab}^{\circledast} = \varnothing$;

(ii') $\langle a, b\rangle \notin R \wedge \langle a, b\rangle \notin Q$ implies $Q \cap R_{ab}^{\circledast} = \varnothing$; and

(iii') $\langle a, b\rangle \notin R \wedge \langle a, b\rangle \notin Q$ implies $Q_{aba} \cap R^{\circledast} = \varnothing$.

To show (i'), by (2), we have $R^{\circledast}_{ab} = R^{\circledast}$, and so $Q \cap R^{\circledast}_{ab} = Q \cap R^{\circledast} = \varnothing$.

To show (ii'), if $\langle b, a\rangle \notin R$, then we have case (i'). Hence, we can assume that $\langle b, a\rangle \in R$. Suppose that $\langle c, d\rangle \in Q \cap (R^{\circledast}_{ab} \setminus R^{\circledast})$. Then, without loss of generality, we may assume that $\langle c, d\rangle \notin R^*$. Hence, by (1), we get $\langle c, a\rangle \in R^*$ and $\langle b, d\rangle \in R^*$. We now consider two cases (see Figure 8).

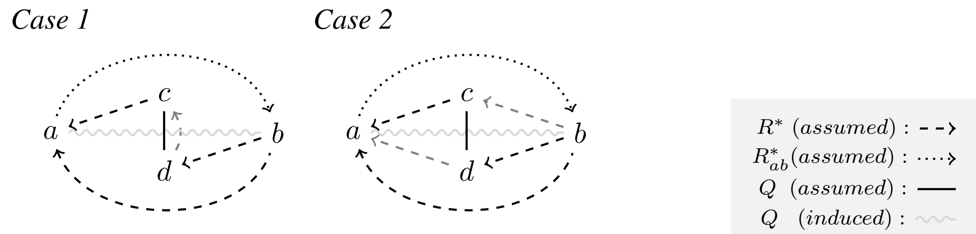

Figure 8. A visualisation of the proof of Lemma 4.2.

Case 1: $\langle d, c\rangle \in R^*$. Then bR^*cdR^*a, and so $\langle a, b\rangle \in Q^{[R]} = Q$, a contradiction.

Case 2: $\langle d, c\rangle \notin R^*$. Then, by $\langle d, c\rangle \in R^*_{ab}$, we have dR^*a and bR^*c. Hence bR^*cdR^*a, and so $\langle a, b\rangle \in Q^{[R]} = Q$, a contradiction.

As a result, $Q \cap R^{\circledast}_{ab} = \varnothing$ since $Q \cap R^{\circledast} = \varnothing$.

To show (iii'), suppose that $\langle c, d\rangle \in Q_{aba} \cap R^{\circledast} \neq \varnothing$. Since $Q \cap R^{\circledast} = \varnothing$, $\langle c, d\rangle$ can only be $\langle a, b\rangle$ or $\langle b, a\rangle$. Without loss of generality, let $\langle c, d\rangle = \langle a, b\rangle$. Then $\langle a, b\rangle \in R^{\circledast}$ implies $\langle a, b\rangle \in R^{\curlywedge} \cap (R^{\curlywedge})^{-1}$, which implies $\langle a, b\rangle \in R$ by Proposition 4.1, a contradiction. □

To complete the properties of the saturation process described in Lemma 4.2 and used in the proof of Theorem 4.1, we formulate the following

Lemma 4.3. Let $gmos = \langle X, Q, R\rangle$ be a GMO-structure such that $a, b \in X$, $a \neq b$, $\langle a, b\rangle \notin R$ and $\langle a, b\rangle \notin Q$ and

$$S' = \langle X, Q, R_{ab}\rangle^{\blacklozenge} = \langle X, Q', R^{\curlywedge}_{ab}\rangle .$$

Then $\langle a, b\rangle \notin Q'$.

Proof:

We first observe that, by Lemma 4.2, S' is GMO-structure. Suppose that $\langle a, b\rangle \in Q'$. If $\langle b, a\rangle \in R^{\curlywedge}_{ab}$, then $\langle a, b\rangle \in R^{\circledast}_{ab}$, contradicting the separability of S'. Hence $\langle b, a\rangle \notin R^{\curlywedge}_{ab}$, and so $\langle b, a\rangle \notin R$. The latter means that $R^{\circledast}_{ab} = R^{\circledast}$. We then consider three cases:

(i) $\langle a, b\rangle \in R^{\circledast} \circ Q \circ R^{\circledast}$;

(ii) $\langle a, b\rangle \in R^{\circledast} \circ (R^*_{ab} \circ_Q R^*_{ab}) \circ R^{\circledast}$; and

(iii) $\langle b, a\rangle \in R^{\circledast} \circ (R^*_{ab} \circ_Q R^*_{ab}) \circ R^{\circledast}$.

If (i) holds then, since $gmos$ is a GMO-structure, we have $\langle a, b\rangle \in Q$, a contradiction.

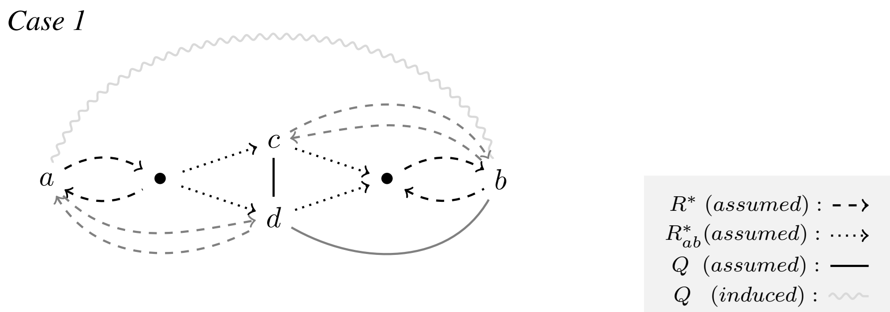

Figure 9. A visualisation of the proof of Lemma 4.3.

If (ii) holds, then there exists $\langle c, d\rangle \in Q$ such that $aR^*_{ab}cdR^*_{ab}b$. Since $\langle a, b\rangle \notin Q$ and $gmos$ is a GMO-structure, aR^*cdR^*b does not hold. Hence, by the symmetry of c and d, we consider two cases (see Figure 9).

Case 1: $\langle a, c\rangle \notin R^*$. Then, by (1), $\langle b, c\rangle \in R^*$. Hence $\langle b, c\rangle \in R^{\circledast}_{ab} = R^{\circledast}$. As a result, by $G5$, we obtain $\langle b, d\rangle \in Q$.
If $\langle a, d\rangle \notin R^*$ then, similarly as in the case of $\langle a, c\rangle \notin R^*$, we have $\langle b, d\rangle \in R^{\circledast}$, which contradicts the separability of $gmos$. Hence $\langle a, d\rangle \in R^*$.
Now, if $\langle d, b\rangle \in R^*$ then $aRdRb$ and dQb. Hence, by $G4$, we have aQb, which contradicts our initial assumption. As a result, $\langle d, b\rangle \notin R^*$. Thus, by (1), $\langle d, a\rangle \in R^*$.
Hence $\langle a, d\rangle \in R^{\circledast}_{ab} = R^{\circledast}$, and so by $G5$ we obtain $\langle a, b\rangle \in Q$, yielding a contradiction with our initial assumption.

Case 2: $\langle d, b\rangle \notin R^*$. Then similarly $\langle a, d\rangle \in R^{\circledast}$ and $\langle b, c\rangle \in R^{\circledast}$, and so $\langle a, b\rangle \in Q$. Summing up, (ii) implies $\langle a, b\rangle \in Q$ and we obtain a contradiction.

If (iii) holds, then $\langle b, a\rangle \in R^*_{ab}$, and we obtain a contradiction with $\langle b, a\rangle \notin R^{\curlywedge}_{ab}$. □

In Lemmas 4.2 and 4.3 we have captured a method of saturating GMO-structures that are not LO-structures. It moreover allows us to formulate an immediate

Corollary 4.2. Every relational structure saturated among all separable relational structures is a layered order structure.

4.2. General concurrent histories

We now return to our original goal to provide a structural characterisation of all histories comprising step sequence executions. Recalling that $\mathsf{os2los}(gmos)$ are the LO-structures associated with a GMO-structure $gmos$, we obtain a new version of Szpilrajn's Theorem:

Theorem 4.1. For every GMO-structure $gmos$,

$$\mathsf{os2los}(gmos) \neq \varnothing \quad \text{and} \quad gmos = \bigcap \mathsf{os2los}(gmos) \ .$$

Proof:
Let $\mathscr{F} = \mathsf{os2los}(gmos)$. The first part is nothing but Corollary 4.2. Let $gmos = \langle X, \rightleftharpoons, \sqsubset \rangle$. We will denote $S = \langle X, \rightleftharpoons_S, \sqsubset_S \rangle$, for any layered extension S of $gmos$.

Since $\mathscr{F}$ is the set of all layered extensions of $gmos$, we know that $gmos \subseteq S$, for all $S \in \mathscr{F}$. Hence $gmos \subseteq \bigcap_{S \in \mathscr{F}} S$. We need to show the reverse inclusion.

We start by proving that $\bigcap_{S \in \mathscr{F}} \sqsubset_S$ is included in $\sqsubset$. Suppose that $a \not\sqsubset b$. We will now define two auxiliary GMO-structures, $gmos'$ and $gmos''$, in the following way. If $a \rightleftharpoons b$ then $gmos' = gmos$. Otherwise,

$$gmos' = \langle X, \rightleftharpoons_{aba}, \sqsubset \rangle^{\blacklozenge} = \langle X, \rightleftharpoons_{aba}^{[\sqsubset]}, \sqsubset \rangle$$

is a GMO-structure, by Lemma 4.2. If $b \sqsubset a$ then $gmos'' = gmos'$. Otherwise,

$$gmos'' = \langle X, \rightleftharpoons_{aba}^{[\sqsubset]}, \sqsubset_{ba} \rangle^{\blacklozenge}$$

is a GMO-structure, by Lemma 4.2. Let $gmos'' = \langle X, \rightleftharpoons'', \sqsubset'' \rangle$.

We have $a \rightleftharpoons'' b \sqsubset'' a$. As a result, for every layered extension S of $gmos''$, we get $a \rightleftharpoons_S b \sqsubset_S a$. Hence, by (8) and Proposition 4.3, we have that $a \not\sqsubset_S b$. Moreover, by $gmos \subseteq gmos' \subseteq gmos''$, each layered extension of $gmos''$ is also a layered extension of $gmos$. Consequently, $\langle a, b \rangle$ is not included in $\bigcap_{S \in \mathscr{F}} \sqsubset_S$, and so the latter is a subset of $\sqsubset$.

Next we show that $\bigcap_{S \in \mathscr{F}} \rightleftharpoons_S$ is included in $\rightleftharpoons$. Suppose that $a \not\rightleftharpoons b$. We will again define two auxiliary GMO-structures, $gmos'$ and $gmos''$, in the following way. If $a \sqsubset b$ then $gmos' = gmos$. Otherwise, by Lemma 4.2,

$$gmos' = \langle X, \rightleftharpoons, \sqsubset_{ab} \rangle^{\blacklozenge}$$

is a GMO-structure. Let $gmos' = \langle X, \rightleftharpoons', \sqsubset' \rangle$. We observe that, by Lemma 4.3, $\langle a, b \rangle \notin \rightleftharpoons'$, hence also $\langle b, a \rangle \notin \rightleftharpoons'$. If $b \sqsubset' a$ then $gmos'' = gmos'$. Otherwise,

$$gmos'' = \langle X, \rightleftharpoons', \sqsubset'_{ba} \rangle^{\blacklozenge}$$

is a GMO-structure, by Lemma 4.2. Let $gmos'' = \langle X, \rightleftharpoons'', \sqsubset'' \rangle$.

We have $a \sqsubset'' b \sqsubset'' a$. As a result, for every layered extension S of $gmos''$, we get $a \sqsubset_S b \sqsubset_S a$. Hence, by (8), we have that $a \not\rightleftharpoons_S b$. Moreover, by $gmos \subseteq gmos' \subseteq gmos''$, each layered extension of $gmos''$ is also a layered extension of $gmos$. Consequently, $\langle a, b \rangle$ is not included in $\bigcap_{S \in \mathscr{F}} \rightleftharpoons_S$, and so the latter is a subset of $\rightleftharpoons$. □

As $\bigcap LOS$ is a GMO-structure, for every nonempty set LOS of LO-structures with the same domain, we can now conclude that all concurrent histories are represented by GMO-structures. Figure 10 shows an example of GMO-closure of an order structure, as well as the set of all the LO-structures extending the resulting GMO-structure.

5. Concluding remarks

We can finally clarify the relationship between GSO-structures and GMO-structures. In general, in order to accept an order structure $os = \langle X, \rightleftharpoons, \sqsubset \rangle$ as an invariant representation of a concurrent history, we require that

$$\mathsf{os2los}(os) \neq \varnothing \quad \text{and} \quad os = \bigcap \mathsf{os2los}(os)\ .$$

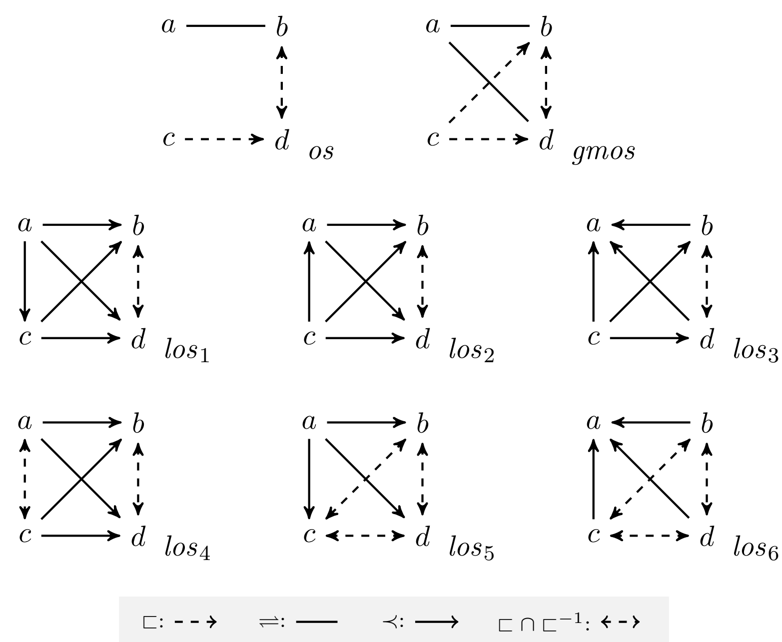

Figure 10. An order structure os; its GMO-closure $gmos = os^{\blacklozenge}$; and the six LO-structures los_1–los_6 extending $gmos$, i.e., $\mathsf{os2los}(gmos) = \{los_1, \ldots, los_6\}$). Note that los_1 corresponds to the step sequence $\{a\}\{c\}\{b,d\}$, los_2 to $\{c\}\{a\}\{b,d\}$, los_3 to $\{c\}\{b,d\}\{a\}$, los_4 to $\{a,c\}\{b,d\}$, los_5 to $\{a\}\{b,c,d\}$, and los_6 to $\{b,c,d\}\{a\}$.

We demonstrated that this property holds whenever os is a GMO-structure, and that it may fail to hold for a GSO-structure. We have further shown that GMO-structures are GSO-structures, but that the converse does not hold. However, each GSO-structure $gsos$ is separable, and so its GMO-closure $gsos^{\blacklozenge}$ is a GMO-structure satisfying $\mathsf{os2los}(gsos^{\blacklozenge}) = \mathsf{os2los}(gsos)$. In other words, concurrent histories described by order structures and their GMO-closures are the same. The importance of GSO-structures comes from the fact that they paved the way for GMO-structures, by exposing the fundamental property that causal ordering is a combination of mutex and weak ordering.

A key motivation for the research presented in this paper comes from concurrent behaviours as exhibited by safe Petri nets with mutex arcs. The resulting semantical approach — which has been meticulously worked out above — is based on GMO-structures which characterise all concurrent histories comprising step sequence executions. A natural direction for further work is to provide a compatible language-theoretic representation of concurrent histories, by generalising Mazurkiewicz traces [17] which correspond to causal posets, and comtraces [10] which correspond to SO-structures (i.e., MO-structures). This development would also allow to link the dynamic notions of mutex and weak causality with the static properties of Petri nets with mutex arcs. The resulting semantics can be regarded as a promising condition for developing more efficient verification techniques [2, 18, 22].

While step sequences (i.e., stratified orders) provide an expressive operational semantics, they still do not represent the most general case. It was argued in [25], and analysed in detail in [9], that the most general observational semantics can be represented by entities equivalent to interval orders. Structures corresponding to stratified order structures that use interval orders instead of step sequences to represent observations have been proposed in [7, 15], and analysed in detail in [11]; however, their extension with a relationship similar to the mutex relationship from this paper is still an open research problem.

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