

INTERLACING PROPERTIES OF TRIDIAGONAL SYMMETRIC MATRICES WITH APPLICATIONS TO PARALLEL COMPUTING

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Abstract. In this paper we present new interlacing properties for the eigenvalues of an unreduced tridiagonal symmetric matrix in terms of its leading and trailing sub matrices. The results stated in Hill and Parlett[8] are hereby improved. The proofs of our results are simple and elementary. We further extend our results to reduced symmetric tridiagonal matrices and to specially structured full symmetric matrices. We then apply these theoretical results to the devising of fast and efficient parallel algorithms for computing the eigenvalues of very large size matrices.

Key Words. Symmetric, Tridiagonal, Eigenvalues, Parallel algorithms.

1. Introduction. The interlacing properties of tridiagonal symmetric matrices are of interest for a variety of algorithms such as the Lanczos tridiagonalization method, and divide and conquer methods for computing the eigenvalues of the matrix. Recently, Hill and Parlett[8] have presented some refined interlacing properties for the eigenvalues of an unreduced tridiagonal symmetric matrix in terms of the eigenvalues of its leading sub matrices. In this paper we generalize their results by providing interlacing properties for the eigenvalues of the matrix in terms of the eigenvalues of the leading and trailing sub matrices, and we show that their results are a special case of ours. Moreover, our exposition is elementary and simple to follow as compared to their's. We further extend our results to general symmetric tridiagonal matrices, and to specially structured full symmetric matrices as well. Finally, we present some applications of our theoretical results to the parallel computation of the eigenvalues of very large size matrices. This paper is organized as follows. In Section 2, we present some definitions and notations. In Section 3, and Section 4, we present our two main Theorems for unreduced symmetric tridiagonal matrices. In Section 5, we extend them to general symmetric and further to other specially structured full symmetric matrices. In Section 6, we present some applications to the parallel computation of the eigenvalues of very large size matrices, and finally in the conclusion we mention some open related problems.

2. Basic definitions and notations. We denote by \mathcal{R}^n , the set of real vectors of order n , and the standard basis for this induced vector space by

$$(1) \quad e_i, \quad i = 1, \dots, n,$$

where e_i is all zeros beside the i th coordinate which is one. When needed we emphasize that a vector is in \mathcal{R}^n by writing for example $e_i^{(n)}$. We denote by $\mathcal{M}(n)$ the set of real matrices of order n . Our main interest in this paper is in the eigenvalues of real symmetric matrices. However, since complex Hermitian matrices have similar properties

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and the extended set of eigenvalues of T_k , by θ , i.e.,

$$(10) \quad \theta = \{\theta_0 = -\infty < \theta_1 < \dots < \theta_k < \infty = \theta_{k+1}\}, \quad 1 \leq k \leq (n-1),$$

where we have added the two extreme artificial eigenvalues for convenience. For $k = n-1$, the eigenvalues of T interlace those of $T_{(n-1)}$, i.e., in each interval

$$(11) \quad (\theta_{i-1}, \theta_i) \quad i = 1, \dots, n,$$

there is exactly one eigenvalue of T . This is just a consequence of the Cauchy's interlacing theorem, see Parlett[10].

3. Interlacing properties for the eigenvalues of an unreduced symmetric tridiagonal matrix. We present in this section our first main result relating the spread of the eigenvalues of the matrix to the spread of the eigenvalues of its leading and trailing sub matrices.

THEOREM 3.1. *Let $T \in \mathcal{M}(n)$ be an unreduced symmetric tridiagonal matrix as in (3). Let θ denotes the extended set of eigenvalues of T_k as in (10), and let,*

$$(12) \quad \beta = \{\beta_0 = -\infty < \beta_1 < \dots < \beta_m < \infty = \beta_{m+1}\}, \quad m = n - (k+1),$$

denotes the extended set of eigenvalues of H_{k+2} . We further let,

$$(13) \quad \gamma = \{\gamma_0 = -\infty < \gamma_1 \leq \dots \leq \gamma_{n-1} < \infty = \gamma_n\},$$

denotes the respective union of θ and β , where by union we distinguish between the same eigenvalues from the two different sets. Then in each interval

$$(14) \quad (\gamma_{i-1}, \gamma_i) \quad i = 1, \dots, n,$$

there is a different eigenvalue, and therefore exactly one different eigenvalue of T .

Remarks:

1. In case $(\gamma_{i-1} = \gamma_i)$ we assume that the interval contains that one point alone.
2. By a different eigenvalue we mean algebraically different, that is, in case the multiplicity of an eigenvalue λ is r , each occurrence is considered to be a different one.
3. The last remark is of no significance here, because the eigenvalues differ by the usual convention. However, when we consider later the more general case of reduced symmetric tridiagonal matrices, an eigenvalue may have any multiplicity up to n . For the sake of consistency we prefer to clarify this definition right at this point.

We will prove Theorem (3.1) in the following:

LEMMA 3.2. *Suppose there exist an index i such that,*

$$(15) \quad \lambda = \gamma_{i-1} = \gamma_i, \quad 2 \leq i \leq (n-1).$$

Then λ is an eigenvalue of T . We note that in this case, γ_{i-1}, γ_i each belongs to a different set, or more formally, there exist indices r and s such that,

$$(16) \quad \theta_r = \gamma_{i-1} = \gamma_i = \beta_s,$$

where $1 \leq r \leq k$, and $1 \leq s \leq m$.

Proof. Consider the characteristic polynomial of T ,

$$(17) \quad p(x) = q_{k+1}(x)p_k(x) - b_k^2 p_{k-1}(x)q_{k+2}(x).$$

Then, for r, s as above,

$$(18) \quad p(\lambda) = q_{k+1}(\lambda)p_k(\theta_r) - b_k^2 p_{k-1}(\lambda)q_{k+2}(\beta_s) = 0.$$

Hence, λ is an eigenvalue of T . \square

LEMMA 3.3. *There exist an index r such that,*

$$(19) \quad \lambda = \theta_r, \quad 1 \leq r \leq k,$$

is an eigenvalue of T , if and only if there exists an index s , such that,

$$(20) \quad \lambda = \beta_s, \quad 1 \leq s \leq m.$$

Proof. We prove one way, the other is proved similarly. Let $\lambda = \theta_r$ be an eigenvalue of T , then,

$$(21) \quad 0 = p(\lambda) = q_{k+1}(\lambda)p_k(\theta_r) - b_k^2 p_{k-1}(\theta_r)q_{k+2}(\lambda) = -b_k^2 p_{k-1}(\theta_r)q_{k+2}(\lambda).$$

However, since T is unreduced $b_k \neq 0$, and since the eigenvalues of T_k strictly interlace those of T_{k-1} we have $p_{k-1}(\theta_r) \neq 0$. Hence, $q_{k+2}(\lambda) = 0$ and λ is an eigenvalue of H_{k+2} . \square

COROLLARY 3.4. *For the proof of Theorem (3.1) it is sufficient to show that there is at most one eigenvalue in each non-redundant interval,*

$$(22) \quad (\gamma_{i-1}, \gamma_i), \quad \gamma_{i-1} < \gamma_i, \quad 1 \leq i \leq n.$$

Proof. For in case of equality,

$$(23) \quad \gamma_{i-2} < \gamma_{i-1} = \gamma_i < \gamma_{i+1},$$

since the eigenvalues of T_k and H_{k+2} are all different. By Lemma (3.2) there is exactly one eigenvalue in each such redundant interval because the eigenvalues of T are all different. By Lemma (3.3) if the endpoint of a non-redundant interval is an eigenvalue of T it is counted for in its respective redundant interval. Hence, if there is at most one eigenvalue in each of the remaining non-redundant intervals, there must be exactly one there, for the total number of eigenvalues is exactly n . \square

We are finally ready to give the proof of Theorem (3.1):

Proof. By the last Corollary we need only prove that there is at most one eigenvalue in each non-redundant interval. For a given index $s, 1 \leq s \leq m+1$, we consider the s th interval of β , that is (β_{s-1}, β_s) . Then there is an index $i, s \leq i \leq (n - (m+1 - s))$, and indices l and r such that,

$$(24) \quad \theta_{r-1} \leq \gamma_{i-1} = \beta_{s-1} < \gamma_i = \theta_r < \dots < \theta_{r+l-1} = \gamma_{i+l-1} < \beta_s = \gamma_{i+l} \leq \theta_{r+l},$$

where $1 \leq r \leq k$, and $0 \leq l \leq (k+1) - r$. We will then show that in each interval,

$$(25) \quad (\gamma_{i+j-1}, \gamma_{i+j}) \quad j = 0, \dots, l,$$

there exist at most one eigenvalue of T . Hence, since for $s = 1, \dots, (m+1)$ these sets of intervals correspond to the complete set of non-redundant intervals of γ we are done. For the proof we make use of Sylvester theorem which states that the inertia of T and F^*TF is the same provided F is non-singular see Horn[9]. We denote the number of positive eigenvalues of T by $\pi(T)$. Given a real number x which is not an eigenvalue of T_k nor of H_{k+2} , we construct the matrix F as follows:

$$(26) \quad F^* = \begin{pmatrix} I_k & 0 & 0 \\ -b_k(e_k^{(k)})^*(T_k - xI)^{-1} & 1 & -b_{k+1}(e_1^{(m)})^*(H_{k+2} - xI)^{-1} \\ 0 & 0 & I_m \end{pmatrix}.$$

Then,

$$(27) \quad \hat{T} - xI = F^*(T - xI)F = \begin{pmatrix} T_k - xI_k & 0 & 0 \\ 0 & \hat{a}_{k+1}(x) - x & 0 \\ 0 & 0 & H_{k+2} - xI_m \end{pmatrix},$$

and therefore

$$(28) \quad \pi(T_k - xI) + \pi(H_{k+2} - xI) \leq \pi(T - xI) \leq \pi(T_k - xI) + \pi(H_{k+2} - xI) + 1.$$

However, the inertias of T_k and H_{k+2} do not change inside a non-redundant interval, as in (25), and therefore the number of eigenvalues of T there is bounded by

$$(29) \quad \pi(T - xI) - \pi(T - yI) \leq 1,$$

for some $x \leq y$ and $x, y \in (\gamma_{i+j-1}, \gamma_{i+j})$. \square

We proceed to reflect on some implications of Theorem (3.1).

- Theorem (3.1) is a generalization of Theorem 1 in Hill and Parlett[8], which states that in each interval

$$(30) \quad (\theta_{i-1}, \theta_i) \quad i = 1, \dots, k + 1,$$

there is at least one different eigenvalue of T . In fact their result is a simple consequence of ours. For let us assume that

$$(31) \quad \gamma_{j-2} < \theta_{i-1} = \gamma_{j-1} \leq \gamma_j \leq \dots \leq \gamma_{j+l-1} \leq \gamma_{j+l} = \theta_i < \gamma_{j+l+1},$$

where $i \leq j \leq n - (k + 1 - i)$, and $0 \leq l \leq n - (k + j + 1 - i)$. (Note that for $j = 1$ we ignore γ_{-1} and for $j + l = n$ we ignore γ_{n+1}). Then there is at least one index r such that,

$$(32) \quad \theta_{i-1} \leq \gamma_{j+r-1} < \gamma_{j+r} \leq \theta_i, \quad 0 \leq r \leq l,$$

because $\theta_{i-1} \neq \theta_i$. Hence, there is at least one eigenvalue of T there. The proof of this statement as given in[8] is long and tedious and involves the inspection of the interlacing properties of four sequences of eigenvalues. Our proof of the much more general result is more simple and straightforward.

- Our bounds are sharper. The number of eigenvalues of T in the i th interval as in (30), is equal exactly to the number of eigenvalues of H_{k+2} that are contained there, plus one. Moreover, if θ_{i-1} or θ_i or both are equal to an eigenvalue of H_{k+2} , then they are respectively also eigenvalues of T .

Proof. Consider the i th interval as in (31), then we observe that,

$$(33) \quad \gamma_{j+r-1} \neq \gamma_{j+r}, \quad r = 1, \dots, l - 1,$$

because they both belong to the set β . Hence,

$$(34) \quad \gamma_{j+r} \in (\theta_{i-1}, \theta_i), \quad r = 1, \dots, l - 2.$$

We conclude that so far, the interval contains $l - 1$ eigenvalues of T , and $l - 2$ eigenvalues of H_{k+2} , which is according to our claim. The remaining two extreme intervals may now increase the size of both of these sets, by the same amount, and the proof now follows. The last assertion is then obvious. \square

- COROLLARY 3.5. *In case*

$$(35) \quad \gamma_{i-1} < \gamma_i < \gamma_{i+1},$$

γ_i is not an eigenvalue of T , and there are exactly i eigenvalues of T before it, and $n - i$ eigenvalues after it. Otherwise, say,

$$(36) \quad \gamma_{i-2} < \gamma_{i-1} = \gamma_i < \gamma_{i+1},$$

γ_i is an eigenvalue of T , and there are exactly $(i - 1)$ eigenvalues before it, and $(n - i)$ eigenvalues after it.

- Theorem (3.1) also implies Theorem 2 of Hill and Parlett which states the following for the special case of $n = k + 2$. Let, $H_n = a_n = a$, and let a belongs to the i th interval of θ , i.e.,

$$(37) \quad a \in [\theta_{i-1}, \theta_i], \quad 1 \leq i \leq k + 1.$$

Then, there is exactly one eigenvalue of T in each of the intervals,

$$(38) \quad (\theta_{j-1}, \theta_j), \quad 1 \leq j \leq k + 1, \quad j \neq i,$$

and in each of the subintervals,

$$(39) \quad (\theta_{i-1}, a), \quad (a, \theta_i).$$

This first part of their theorem is the conclusion of ours. Next, consider the extended set of eigenvalues of T_{k+1} which we denote by χ , i.e.,

$$(40) \quad \chi = \{\chi_0 = -\infty < \chi_1 < \dots < \chi_{k+1} < \infty = \chi_{k+2}\}.$$

Note that from the Cauchy's interlace theorem the eigenvalues of χ strictly interlace those of θ . Let, $a = \chi_i$, i.e.,

$$(41) \quad \chi_{i-1} < \theta_{i-1} < \chi_i = a < \theta_i < \chi_{i+1}.$$

Then, beside the two eigenvalues of T which lie in (39), i.e., in

$$(42) \quad (\theta_{i-1}, \chi_i), \quad (\chi_i, \theta_i),$$

there is exactly one eigenvalue in each sub interval,

$$(43) \quad (\theta_{j-1}, \chi_j), \quad j = 1, \dots, (i - 1),$$

and in each sub interval

$$(44) \quad (\chi_j, \theta_j), \quad j = (i + 1), \dots, (k + 1).$$

This is the essence of the second part of their theorem, and is again a simple consequence of ours. For by the Cauchy's interlacing theorem, there is exactly one eigenvalue of T in each of the intervals,

$$(45) \quad (\chi_{j-1}, \chi_j), \quad 1 \leq j \leq k+2,$$

and in particular exactly one in the i th and $(i+1)$ th intervals. Hence, there can be no eigenvalue of T in the subintervals

$$(46) \quad (\chi_{i-1}, \theta_{i-1}), \quad (\theta_i, \chi_{i+1}),$$

and the conclusion in (43) and (44) now follows. In case $\chi_i \neq a$, the theorem further bounds the eigenvalues of T in the i th interval of θ above as follows. For $\chi_i < a$ there is exactly one eigenvalue of T in the sub intervals,

$$(47) \quad (\theta_{i-1}, a), \quad (\chi_i, \theta_i),$$

and for $a < \chi_i$ there is exactly one eigenvalue in the sub intervals,

$$(48) \quad (\theta_{i-1}, \chi_i), \quad (a, \theta_i).$$

These conclusions are again an obvious result of our Theorem. However, we will give a much more general result in the next section for which this theorem of Hill and Parlett is a special case.

- Theorem (3.1) slightly modified also holds for general tridiagonal matrices. However, for the sake of simplicity we have decided to consider first this more easily verifiable case. The general case is dealt with in Section 5.
- There are some generalizations of Theorem (3.1) to specially structured full symmetric matrices. These will be considered also in Section 5.
- There are some consequences of our theoretical results to the parallel computation of the eigenvalues of very large size matrices. For example, by choosing $k = \lfloor (n-1)/2 \rfloor$, we can compute the eigenvalues of T_k and H_{k+2} in parallel, and then use their interlacing properties to get some sharper bounds for the exact eigenvalues of T . For example, consider the famous tridiagonal matrix $T = \text{tridiag}(-1, 2, -1)$, whose eigenvalues are given analytically by,

$$(49) \quad \lambda_i = 4 \sin^2 \frac{i\pi}{2(n+1)}, \quad i = 1, \dots, n.$$

Let $n = 1024$, and assume for our demonstration that we are looking for the 307th and 308th eigenvalues which are depicted below,

$$(50) \quad \lambda = \left(\lambda_{307} = 0.82195126525, \quad \lambda_{308} = 0.82691048642 \right).$$

For $k = 511$, and $m = 512$, the related eigenvalues of θ and β are,

$$(51) \quad \theta = \left(\theta_{153} = 0.81848059628, \quad \theta_{154} = 0.82840428509 \right),$$

and

$$(52) \quad \beta = \left(\beta_{153} = 0.81552949467, \quad \beta_{154} = 0.82542059370 \right).$$

We conclude that,

$$(53) \quad \lambda_{307} \in \left(\theta_{153}, \beta_{154} \right) = \left(0.81848059628, 0.82542059370 \right),$$

and that

$$(54) \quad \lambda_{308} \in \left(\beta_{154}, \theta_{154} \right) = \left(0.82542059370, 0.82840428509 \right).$$

In fact, we have been able to isolate each of these eigenvalues in a subinterval containing that eigenvalue alone. Thereafter, we may use fast iterative methods such as the QR algorithm, to locate that eigenvalue more accurately. We elaborate more on these applications in Section 6.

4. Refined interlacing properties for the eigenvalues of an unreduced symmetric tridiagonal matrix. We present in this section more refined results relating the spread of the eigenvalues of a matrix to the eigenvalues of a two leading and a two trailing sub matrices.

THEOREM 4.1. *Let $T \in \mathcal{M}(n)$ be an unreduced symmetric tridiagonal matrix as in (3). Let θ denotes the extended set of eigenvalues of T_k as in (10), let β denotes the extended set of eigenvalues of H_{k+2} as in (12), and let γ denotes their respective union as in (13). Similarly, let ϕ ,*

$$(55) \quad \phi = \{\phi_0 = -\infty < \phi_1 < \dots < \phi_{k-1} < \infty = \phi_k\},$$

denotes the extended set of eigenvalues of T_{k-1} , let α ,

$$(56) \quad \alpha = \{\alpha_0 = -\infty < \alpha_1 < \dots < \alpha_{m+1} < \infty = \alpha_{m+2}\},$$

denotes the extended set of eigenvalues of H_{k+1} , and let δ ,

$$(57) \quad \delta = \{\delta_0 = -\infty < \delta_1 \leq \dots \leq \delta_{n-1} < \infty = \delta_n\},$$

denotes the respective union of ϕ and α . Finally, let ϵ ,

$$(58) \quad \epsilon = \{\epsilon_0 = -\infty < \epsilon_1 \leq \dots \leq \epsilon_{2(n-1)} < \infty = \epsilon_{2n-1}\},$$

denotes the respective union of the sets γ and δ . Then, in each interval,

$$(59) \quad (\epsilon_{2i}, \epsilon_{2i+1}) \quad i = 0, \dots, (n-1),$$

there is a different eigenvalue, and therefore exactly one different eigenvalue of T .

Proof. We will first assume for simplicity that ϵ do not contain any duplicates, and that w.l.g. $\delta_1 < \gamma_1$. The proof is by induction on the index i , as in (59).

For $i = 0$. The interval

$$(60) \quad (\epsilon_0, \epsilon_1) \equiv (-\infty, \delta_1)$$

indeed contains exactly one different eigenvalue of T by Theorem (3.1).

For $i > 0$, we will denote by,

$$(61) \quad \Sigma_{i-1} \equiv (\epsilon_{2(i-1)}, \epsilon_{2i-1}),$$

the last interval to contain an eigenvalue of T , i.e.,

$$(62) \quad \Sigma_0 \equiv (\gamma_0, \delta_1).$$

Note, that it is sufficient to show that there is at least one eigenvalue in each interval, because these intervals are disjoint and the total number of eigenvalues can not exceed n .

Let $\Sigma_{i-1} \equiv (\gamma_{i-1}, \delta_i)$.

Then, $\epsilon_{2i} = \gamma_i$, for otherwise there will be two eigenvalues in the interval (γ_{i-1}, γ_i) , a contradiction to Theorem (3.1). Therefore, either,

$$(63) \quad \Sigma_i \equiv (\gamma_i, \delta_{i+1}),$$

or

$$(64) \quad \Sigma_i \equiv (\gamma_i, \gamma_{i+1}),$$

and both cases are valid. In case (63) holds, the eigenvalue of (γ_{i-1}, γ_i) , is already in Σ_{i-1} above, and there can be no eigenvalue in (δ_i, γ_i) . However, by Theorem (3.1) there is also exactly one eigenvalue of T in (δ_i, δ_{i+1}) , and since by Lemma(3.3) this can not be γ_i , it is in Σ_i . In case (64) holds, there is certainly an eigenvalue of T in Σ_i by Theorem (3.1).

Let $\Sigma_{i-1} \equiv (\gamma_{i-1}, \gamma_i)$.

Since ϵ do not contain duplicates, by Corollary (3.5) $\delta_{i-1} < \gamma_i$, and $\gamma_{i-1} < \delta_i$. Hence, $\epsilon_{2i} = \delta_i$ and either

$$(65) \quad \Sigma_i \equiv (\delta_i, \delta_{i+1}),$$

or

$$(66) \quad \Sigma_i \equiv (\delta_i, \gamma_{i+1}),$$

and both cases are valid. The justification for this last statement is similar to that for the first case and is left for the reader.

We further let the reader verify for himself that the two remaining cases are

$$(67) \quad \Sigma_{i-1} \equiv (\delta_{i-1}, \delta_i),$$

and

$$(68) \quad \Sigma_{i-1} \equiv (\delta_{i-1}, \gamma_i),$$

and that they are dealt with in a similar way.

In order to prove the theorem for the more general case we observe the following:

1. In case γ and δ do not contain duplicates but for some i and j ,

$$(69) \quad \gamma_i = \delta_j, \quad 0 \leq i, j \leq n,$$

then $i = j$, and γ_i is not an eigenvalue of T . This is a consequence of Lemma (3.3) and Corollary (3.5).

2. In case γ or δ contains some duplicates, for example,

$$(70) \quad \gamma_{i-2} < \gamma_{i-1} = \lambda_i = \gamma_i < \gamma_{i+1},$$

then λ_i is an eigenvalue of T , and

$$(71) \quad \gamma_{i-2} < \delta_{i-1} < \gamma_{i-1} = \gamma_i < \delta_i < \gamma_{i+1}.$$

A similar observation holds for the case when δ has duplicates. In this example, $\delta_{i-1} \neq \gamma_{i-1}$ for otherwise $\delta_{i-1} = \gamma_{i-1} = \gamma_i = \delta_i$, by Lemma (3.3) and Corollary (3.5), a contradiction to the strict interlacing properties of the sets ϕ, θ and β, α . The rest follows again from Corollary (3.5).

3. By the first observation $\gamma_i = \delta_i$ is not an eigenvalue of T . Hence,

$$(72) \quad \epsilon_{2i-2} < \epsilon_{2i-1} = \gamma_i = \delta_i = \epsilon_{2i} < \epsilon_{2i+1},$$

and we may assume w.l.g. that $\gamma_i < \delta_i$. By the second observation, if $\lambda_i = \gamma_{i-1} = \gamma_i$ is an eigenvalue of T then,

$$(73) \quad \epsilon_{2i-3} = \delta_{i-1} < \epsilon_{2i-2} = \gamma_{i-1} = \gamma_i = \epsilon_{2i-1} < \epsilon_{2i} = \delta_i,$$

and we may assume w.l.g. that $\gamma_{i-1} < \gamma_i$.

We therefore conclude that the same proof is valid for the general case too. \square

We proceed to reflect on some of the implications of Theorem (4.1).

- Theorem (4.1) is a generalization of Theorem 2 in Hill and Parlett[8], which is reviewed for convenience in the fourth remark after Theorem (3.1). For let $n = k + 2$, as is there, and choose $k' = n - 1$. Then applying Theorem (4.1) with k' , the eigenvalues of $T_{k'-1}$, are the eigenvalues in θ , the eigenvalues of T_k are those in χ , $H_{k'+1} = a_{k+2} = a$, and $H_{k'+2}$ is empty. The result now follows word by word from the conclusion of Theorem (4.1), and is much simpler than the previous proof based on Theorem (3.1) which is already a simplification of the proof given in[8].
- Our result is much more general than the one given in[8] since it applies to any sub matrix of order $1 \leq k \leq n$, and not just for the case where $k = n - 2$.
- Theorem (4.1) slightly modified also holds for general tridiagonal matrices. However, as before, for the sake of simplicity we have decided to consider first this more easily verifiable case. The more general case will be considered in Section 5.
- Theorem (4.1) sheds more light on the location of the eigenvalues of T . Consider the previous example with $n = 1024$, and $\lambda_{307}, \lambda_{308}$ as in (50). Setting $k - 1 = 510$, and $m + 1 = 513$, we have computed the related intervals of ϕ and α below,

$$(74) \quad \phi = \left(\phi_{153} = 0.82144722828, \quad \phi_{154} = 83140365274 \right),$$

and

$$(75) \quad \alpha = \left(\alpha_{154} = 0.82245247396, \quad \alpha_{155} = 0.83235511804 \right).$$

Hence, we conclude that,

$$(76) \quad \lambda_{307} \in \left(\phi_{153}, \alpha_{154} \right) = \left(0.82144722828, 0.82245247396 \right)$$

improving the approximation from about 0.0035 in (53), to about 0.0005 above. The approximation for λ_{308} remains the same. Note, that the additional information can be obtained in parallel, using several processors.

- It seems that by taking more and more complementary matrices, we may get even better bounds. But, it is questionable whether this computationally beneficial in practice. We will therefore not pursue this subject any further.

5. Generalizations. We extend Theorems (3.1) and (4.1) of the previous Sections for general tridiagonal symmetric matrices in sub Section 5.1. and sub Section 5.2. Then, in sub Section 5.3 we give some generalizations to specially structured full symmetric matrices.

5.1. Symmetric tridiagonal matrices I. We extend Theorem (3.1) for general symmetric tridiagonal matrices in the following:

THEOREM 5.1. *Let $T \in \mathcal{M}(n)$ be a symmetric tridiagonal matrix as in (3). Let θ ,*

$$(77) \quad \theta = \{\theta_0 = -\infty < \theta_1 \leq \dots \leq \theta_k < \infty = \theta_{k+1}\}, \quad 1 \leq k \leq (n-1),$$

denotes the extended set of eigenvalues of T_k , and let β

$$(78) \quad \beta = \{\beta_0 = -\infty < \beta_1 \leq \dots \leq \beta_m < \infty = \beta_{m+1}\}, \quad m = n - (k+1),$$

denotes the extended set of eigenvalues of H_{k+2} . Let γ denotes their respective union as in (13). Then in each interval

$$(79) \quad [\gamma_{i-1}, \gamma_i] \quad i = 1, \dots, n,$$

we can choose a different eigenvalue of T in a unique way.

Remarks:

1. We may have $\gamma_{i-1} < \gamma_i$, and yet γ_{i-1} or γ_i or both are eigenvalues of T . This is why we must use the closed parenthesis notation.
2. Let $\hat{\lambda}$ corresponds to the sequence of eigenvalues thus chosen. Then we say that $\hat{\lambda}$ is a legal sequence.
3. By uniqueness we mean that if $\hat{\lambda}$ is a legal sequence, then,

$$(80) \quad \hat{\lambda}_i = \lambda_i, \quad i = 1, \dots, n.$$

Proof. We first prove existence and then uniqueness. We show that there is at least one different eigenvalue in each such interval.

Case $b_k, b_{k+1} \neq 0$. For some index $1 \leq k_1 \leq k$, and index $(k+2) \leq k_2 \leq n$,

$$(81) \quad T_k = \begin{pmatrix} T_{1, (k_1-1)} & 0 \\ 0 & T_{k_1, k} \end{pmatrix}, \quad H_{k+2} = \begin{pmatrix} H_{k+2, k_2} & 0 \\ 0 & H_{k_2+1} \end{pmatrix},$$

where $T_{k_1, k}$ and H_{k+2, k_2} are unreduced. We further denote by \hat{T} the unreduced matrix,

$$(82) \quad \hat{T} = \begin{pmatrix} T_{k_1, k} & b_k & & & \\ & b_k & a_{k+1} & b_{k+1} & \\ & & & & H_{k+2, k_2} \\ & & b_{k+1} & & \end{pmatrix}.$$

Let $\hat{\theta} \subset \theta$ denotes the eigenvalues of $T_{k_1, k}$, and let $\hat{\beta} \subset \beta$ denotes the eigenvalues of H_{k+2, k_2} . Let $\hat{\gamma} \subset \gamma$ denotes the union of these respective two sets. Applying

Theorem (3.1) to \hat{T} , with $k' = k - (k_1 - 1)$ and $n' = k_2 - (k_1 - 1)$, we conclude that there is exactly one different eigenvalue of \hat{T} in each interval

$$(83) \quad \hat{\lambda}_i \in (\hat{\gamma}_{i-1}, \hat{\gamma}_i), \quad i = 1, \dots, k_2 - k_1 + 1.$$

However, the eigenvalues of \hat{T} are also eigenvalues of T , and the remaining eigenvalues of T , namely those of $T_{1,(k_1-1)}$, and of H_{k_2+1} , are simply the eigenvalues in the set $\gamma - \hat{\gamma}$. We next describe how we choose a different eigenvalues from each interval of (79) that is a subinterval of the same interval of (83). Since these last intervals are disjoint, and cover the whole real line, the proof then follows. Given an index i , $1 \leq i \leq k_2 - k_1 + 1$, there are indices l, r, s such that,

$$(84) \quad \begin{aligned} \gamma_{l-2} < \hat{\gamma}_{i-1} = \gamma_{l-1} \\ &\leq \gamma_l \leq \dots \leq \gamma_{l+r-1} \leq \hat{\lambda}_i < \gamma_{l+r} \leq \dots \leq \gamma_{l+s-1} < \\ &\gamma_{l+s} = \hat{\gamma}_i \leq \gamma_{l+s+1}, \end{aligned}$$

where $1 \leq i \leq l \leq n - (k_2 + 1 - k_1 - i)$, and $0 \leq r \leq s \leq n - (k_2 + l + 1 - k_1 - i)$. (Note that for $l = 1$ we omit γ_{l-2} , and for $n = (l + s)$ we omit γ_{l+s+1} .) We now choose,

$$(85) \quad \gamma_{l+j} \in (\gamma_{l+j-1}, \gamma_{l+j}], \quad j = 0, \dots, (r - 1),$$

$$(86) \quad \hat{\lambda}_i \in [\gamma_{l+r-1}, \gamma_{l+r}),$$

$$(87) \quad \gamma_{l+j-1} \in [\gamma_{l+j-1}, \gamma_{l+j}), \quad j = (r + 1), \dots, s.$$

which is indeed a correct choice.

Case $b_k = 0, b_{k+1} \neq 0$: Here, $\hat{T} = H_{k+1,k_2}$ and $k' = 0, n' = k_2 - k$ in the notations above. The same proof then holds. The case $b_k \neq 0, b_{k+1} = 0$ is similar.

Case $b_k = b_{k+1} = 0$: The eigenvalues of T are those of γ together with a_{k+1} . The proof is now trivial.

This end the proof for the existence of a legal sequence. The uniqueness follows from Corollary (5.2) below. \square

COROLLARY 5.2. *For any legal sequence $\hat{\lambda}$, we must have,*

$$(88) \quad \hat{\lambda}_i = \lambda_i, \quad i = 1, \dots, n,$$

and therefore λ is a legal sequence.

Proof. Let $1 \leq i \leq n$ be the minimal index such that $\hat{\lambda}_i \neq \lambda_i$. By the minimality of i ,

$$(89) \quad \hat{\lambda}_i = \lambda_j, \quad i < j \leq n,$$

and therefore, $\lambda_i < \hat{\lambda}_i \leq \gamma_i$. However, in that case, there is no way to choose λ_i in the subsequent intervals, and this is a contradiction as $\hat{\lambda}$ is a legal sequence. \square

COROLLARY 5.3. *Consider the case where, for some index $1 \leq l \leq n$,*

$$(90) \quad \gamma_{l-2} < \gamma_{l-1} = \cdots = \gamma_{l+r-1} < \gamma_{l+r}.$$

Then λ_l is an eigenvalue of T of multiplicity at least r but of no more than $r + 2$.

Proof. We first observe that by Corollary (5.2),

$$(91) \quad \lambda_l = \lambda_j \in [\gamma_{j-1}, \gamma_j], \quad j = l, \dots, l + r - 1,$$

and therefore its multiplicity is at least r . The rest follows from the proof of Theorem (5.1). For example, consider the case where $b_k, b_{k+1} \neq 0$. Then, if there exist an index $i, 1 \leq i \leq k_2 - k_1 + 1$ such that

$$(92) \quad \hat{\gamma}_{i-1} = \gamma_{l-1} \quad \text{or} \quad \gamma_{l+r-1} = \hat{\gamma}_i,$$

then the multiplicity of λ_l is at most $r + 1$, and otherwise, for some index i as above,

$$(93) \quad \hat{\gamma}_{i-1} < \gamma_{l-1} \quad \text{and} \quad \gamma_{l+r-1} < \hat{\gamma}_i,$$

and its multiplicity is at least $r + 1$ and at most $r + 2$. \square

5.2. Symmetric tridiagonal matrices II. We proceed to extend Theorem (4.1) for general symmetric tridiagonal matrices in the following:

THEOREM 5.4. *Let $T \in \mathcal{M}(n)$ be a symmetric tridiagonal matrix as in (3), and let θ, β , and γ be as in Theorem (5.1). Similarly, let ϕ ,*

$$(94) \quad \phi = \{\phi_0 = -\infty < \phi_1 \leq \cdots \leq \phi_{k-1} < \infty = \phi_k\}, \quad 1 \leq k \leq (n - 1),$$

denotes the extended set of eigenvalues of T_{k-1} , let α ,

$$(95) \quad \alpha = \{\alpha_0 = -\infty < \alpha_1 \leq \cdots \leq \alpha_{m+1} < \infty = \alpha_{m+2}\}, \quad m = n - (k + 1),$$

denotes the extended set of eigenvalues of H_{k+1} , and let δ denotes the respective union of ϕ and α . Finally, let ϵ denotes the respective union of γ and δ as in Theorem (4.1). Then, in each interval,

$$(96) \quad [\epsilon_{2i}, \epsilon_{2i+1}] \quad i = 0, \dots, (n - 1),$$

we can choose a different eigenvalue of T in a unique way.

Proof. The proof is similar to the proof of Theorem (5.1).

Case $b_{k-1}, b_k, b_{k+1} \neq 0$: Here, we apply Theorem (4.1) to the subsequence $\hat{\epsilon}$ corresponding to the unreduced matrix \hat{T} as in (82). Then, we add the remaining eigenvalues in $\epsilon - \hat{\epsilon}$ which are also eigenvalues of T . However, we note that this time each such eigenvalue of T , distinguishing between the different occurrences of the same eigenvalue, appears exactly twice. The rest now follows by a straightforward modification of the proof of Theorem (5.1).

Case $b_{k-1} = 0, b_k, b_{k+1} \neq 0$: This case is similar to the previous one where we take

$$(97) \quad \hat{T} = \begin{pmatrix} a_k & b_k \\ b_k & H_{k+1, k_2} \end{pmatrix},$$

and $k' = 1, n' = k_2 - (k - 1)$. The case $b_{k-1}, b_k \neq 0$ but $b_{k+1} = 0$ is dealt with in a similar way.

Case $b_{k-1} = b_k = b_{k+1} = 0$: is trivial, and is left for the reader.

Case $b_k = 0, b_{k-1}, b_{k+1} \neq 0$: By the previous discussion we may assume w.l.g that T_k and H_{k+1} are unreduced. We will denote by,

$$(98) \quad \epsilon^{(1)} = \{\phi_0 = -\infty < \theta_1 < \phi_1 \cdots < \phi_{k-1} < \theta_k < \infty = \phi_k\},$$

the respective union of ϕ and θ , and by

$$(99) \quad \epsilon^{(2)} = \{\beta_0 = -\infty < \alpha_1 < \beta_1 \cdots < \beta_m < \alpha_{m+1} < \infty = \beta_{m+1}\},$$

the respective union of β and α . We will then show how to choose a different eigenvalue from each interval in (96) that is a subinterval of the i th interval of ϕ for $1 \leq i \leq k$. Since these intervals are disjoint, and cover the whole real line the proof will be complete. For a given index i , let l, r and s be the indices such that,

$$(100) \quad \phi_{i-1} < \phi_{i-1} \leq \alpha_l < \cdots < \alpha_{l+r-1} \leq \theta_i < \alpha_{l+r} < \cdots < \alpha_{l+s-1} < \phi_i \leq \alpha_{l+s},$$

where $1 \leq l \leq (m+2)$ and $0 \leq r \leq s \leq (m+2-l)$. (Here we further assume by convention that $-\infty < -\infty$ and that $\infty \leq \infty$.) Then in case $\phi_{i-1} \leq \beta_{l-1} < \phi_i$, we choose,

$$(101) \quad \alpha_j \in (\beta_{j-1}, \alpha_j], \quad j = l, \dots, (l+r-1),$$

$$(102) \quad \theta_i \in (\beta_{l+r-1}, \theta_i], \quad \text{or} \quad \theta_i \in [\theta_i, \beta_{l+r-1}),$$

$$(103) \quad \alpha_j \in [\alpha_j, \beta_j), \quad j = (l+r), \dots, (l+s-2),$$

and

$$(104) \quad \alpha_{l+s-1} \in [\alpha_{l+s-1}, \beta_{l+s-1}) \quad \beta_{l+s-1} < \phi_i,$$

or,

$$(105) \quad \alpha_{l+s-1} \in [\alpha_{l+s-1}, \phi_i) \quad \phi_i \leq \beta_{l+s-1}.$$

Otherwise, for $\phi_i \leq \beta_{l-1}$, we choose only,

$$(106) \quad \theta_i \in [\theta_i, \phi_i),$$

and for $\beta_{l-1} < \phi_{i-1}$, we choose,

$$(107) \quad \alpha_l \in (\phi_{i-1}, \alpha_l], \quad \alpha_l \leq \theta_i,$$

or,

$$(108) \quad \theta_i \in (\phi_{i-1}, \theta_i], \quad \theta_i < \alpha_l,$$

and the rest of the respective intervals in (101) to (105). In order to prove our claim we have to show that starting from the first interval of ϵ we choose a different eigenvalue in every second interval. For that purpose, we note that in each interval of ϕ , starting from the first or second subinterval, we do choose a different eigenvalue in every second subinterval. Hence, our strategy is correct if we could show that we choose a different eigenvalue from the first subinterval of the first interval of ϕ and that the transition between consecutive intervals of ϕ abides to this same rule. We will prove that this is so by induction on i .

For $i = 1$: We have,

$$(109) \quad [\epsilon_0, \epsilon_1] = [\phi_0, \alpha_1] \quad \text{or} \quad [\phi_0, \theta_1],$$

and by convention,

$$(110) \quad -\infty = \beta_0 < \phi_0 = -\infty,$$

so that case (107) or (108) holds. Hence, we do choose a different eigenvalue from the first interval of ϵ .

We may now assume by induction that our strategy is correct for up to and not including the i th $i > 0$ interval.

For $i > 0$: Consider the relative position of β_{l-1} in (100). Then, in case $\beta_{l-1} < \phi_{i-1}$, we observe by (104) that we end choosing an eigenvalue from the $(i-1)$ th interval of ϕ in the next to last subinterval. Furthermore, by (107) or (108) we start choosing an eigenvalue from the first subinterval of the i th interval of ϕ . Hence, the transition abides to that rule. Similarly, in case $\phi_{i-1} \leq \beta_{l-1}$, we observe by (105) that we end choosing an eigenvalue from the $(i-1)$ th interval of ϕ in the last subinterval, and that by

(101) or (102) or (106) we start choosing an eigenvalue from the second subinterval of the i th interval of ϕ . Hence, the transition abides to that rule in this case too.

This ends the proof for existence. Uniqueness then follows as from Corollary (5.2). \square

5.3. Specially structured full symmetric matrices. We will present in this subsection some generalizations of the results of the previous subsections to full symmetric matrices of a special structure.

COROLLARY 5.5. *Let $A \in \mathcal{M}(n)$ be a symmetric matrix as follows,*

$$(111) \quad A = \begin{pmatrix} A_k & v_k & \\ v_k^* & a_{k+1} & w_m^* \\ & w_m & B_{k+2} \end{pmatrix} \quad \begin{array}{l} A_k \in \mathcal{M}(k), \quad B_{k+2} \in \mathcal{M}(m), \\ v_k \in \mathcal{R}^k, \quad w_m \in \mathcal{R}^m, \\ m = n - (k + 1). \end{array}$$

Let us denote the extended set of eigenvalues of A_k by θ as in (77), and the extended set of eigenvalues of B_{k+2} by β as in (78). Let γ denotes their respective union as in (13). Then in each interval

$$(112) \quad [\gamma_{i-1}, \gamma_i], \quad i = 1, \dots, n,$$

we can choose a different eigenvalue of A in a unique way.

Proof. We will show that A is similar to a tridiagonal matrix T as in (3) such that,

$$(113) \quad \theta(A_k) = \theta(T_k), \quad \beta(B_{k+2}) = \beta(H_{k+2}).$$

The rest then follows from Theorem (5.1). Let, $V_k \in \mathcal{M}(k)$ and $U_m \in \mathcal{M}(m)$, be orthogonal matrices such that

$$(114) \quad U_m^* w_m = b_{k+1} e_1^{(m)}, \quad V_k^* v_k = b_k e_k^{(k)}, \quad Q = \begin{pmatrix} V_k & & \\ & 1 & \\ & & U_m \end{pmatrix},$$

where Q is orthogonal. Then,

$$(115) \quad \hat{A} = Q^* A Q = \begin{pmatrix} \hat{A}_k & b_k & \\ b_k & a_{k+1} & b_{k+1} \\ & b_{k+1} & \hat{B}_{k+2} \end{pmatrix} \quad \begin{array}{l} \hat{A}_k = V_k^* A_k V_k, \\ \hat{B}_{k+2} = U_m^* B_{k+2} U_m, \end{array}$$

and \hat{A}_k, A_k as well as \hat{B}_{k+2}, B_{k+2} are similar. Finally by a bottom-up tridiagonalization of \hat{A}_k and by a top-down tridiagonalization of \hat{B}_{k+2} , we obtain the tridiagonal matrix T similar to \hat{A} . \square

COROLLARY 5.6. *Let $A \in \mathcal{M}(n)$ be a symmetric matrix as follows,*

$$(116) \quad A = \begin{pmatrix} A_k & C_{mk} \\ C_{mk}^* & B_{k+1} \end{pmatrix} \quad \begin{array}{l} B_{k+1} \in \mathcal{M}(m+1), \\ m = (n - (k+1)), \end{array}$$

where C_{mk} is a rank one matrix. Let θ be as before. Then in each interval

$$(117) \quad [\theta_{i-1}, \theta_i] \quad i = 1, \dots, (k+1),$$

there is at least one different eigenvalue of A . Similarly, let α denotes the extended set of eigenvalues of B_{k+1} as in (95). Then also in each interval

$$(118) \quad [\alpha_{i-1}, \alpha_i] \quad i = 1, \dots, (m+2),$$

there is at least one different eigenvalue of A .

Proof. Since C_{mk} is a rank one matrix, there exist orthogonal matrices $V_k \in \mathcal{M}(k)$ and $U_{m+1} \in \mathcal{M}(m+1)$, such that

$$(119) \quad U_{m+1}^* C_{mk}^* V_k = \begin{pmatrix} 0 & b_k \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} V_k & 0 \\ 0 & U_{m+1} \end{pmatrix},$$

where Q is orthogonal. Hence,

$$(120) \quad \hat{A} = Q^* A Q = \begin{pmatrix} \hat{A}_k & b_k \\ b_k & \hat{B}_{k+1} \end{pmatrix} \quad \begin{array}{l} \hat{A}_k = V_k^* A_k V_k, \\ \hat{B}_{k+1} = U_{m+1}^* B_{k+1} U_{m+1}, \end{array}$$

and \hat{A}_k, A_k as well as \hat{B}_{k+1}, B_{k+1} are similar. Finally \hat{A} is similar to a tridiagonal matrix T as before, and the proof now follows from Theorem (5.1). \square

We note in passing that this is the same as Theorem 4 in Hill and Parlett[8]. However, we can say much more in the following.

COROLLARY 5.7. *Let $A \in \mathcal{M}(n)$ be a symmetric matrix as in (116), and let η denotes the sequence of mixed eigenvalues of θ and α , then in each interval*

$$(121) \quad (\eta_{i-1}, \eta_i), \quad \eta_{i-1} < \eta_i, \quad i = 1, \dots, (n+1),$$

there are either no eigenvalue, one simple eigenvalue, or two simple eigenvalues of A .

Proof. We consider the intervals in (121) that belong to a given interval of α . Let, $1 \leq s \leq (m+2)$, then there exist indices i, l and r , such that

$$(122) \quad \theta_{r-1} < \eta_{i-1} = \alpha_{s-1} \leq \eta_i = \theta_r \leq \dots \leq \theta_{r+l-1} = \eta_{i+l-1} < \alpha_s = \eta_{i+l} \leq \theta_{r+l}.$$

In case there is some index t such that,

$$(123) \quad \beta_{s-1} = \eta_{i+t-1}, \quad 0 \leq t \leq l+1,$$

Step $s = 1, \dots, t$: We find for each sub matrix,

$$(130) \quad T_i^s \quad i = 1, \dots, p/2^s,$$

an interval enclosing only the two eigenvalues of T_i^s which are closer to x from both sides, i.e., $x \in \mathcal{I}_i^s$.

Implementation: We use the information from the previous step. We first merge the corresponding intervals, i.e.,

$$(131) \quad \hat{\mathcal{I}}_i^s = \mathcal{I}_{2i-1}^{s-1} \cup \mathcal{I}_{2i}^{s-1},$$

which from Theorem (3.1) must contain at least three eigenvalues of T . We then use the fast parallel bisection algorithm of Bar-On[1, 3], or the fast parallel QR algorithm of Bar-On[2, 4], or of Bar-On and Codenotti[5, 6] to locate the corresponding interval. We note that as we already have a sharp bound on this interval, we should expect that this procedure would require only very few iterations.

Step $s = t + 1$: Having located a sharp bound for the enclosing eigenvalues near x , we may use again one of the fast parallel algorithms above to locate them more accurately.

We note that the parallel algorithm so presented requires approximately $O(\log(p))$ iterations, and as thus it may even outperform straight sequential strategies of Bisection and QR . Furthermore, the algorithm can be extended to the finding of few eigenvalues in a given Interval, or all the eigenvalues of the matrix. In this respect there are some strong connections between our algorithm and Cuppen's Divide and Conquer[7] method which require further investigations.

7. Conclusion. We have presented new theoretical results relating the eigenvalues of a tridiagonal symmetric matrix to those of its leading and trailing sub matrices. These theoretical results were also generalized to specially structured full symmetric matrices. We have then applied these results and obtained fast and efficient parallel algorithms for locating the eigenvalues of very large size matrices. Further research is still required for the investigations of related results for specially structured sparse symmetric matrices.

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